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# Exact geometric theory for flexible, fluid-conducting tubes

## *Théorie géométriquement exacte des tuyaux souples avec écoulement interne*

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ABSTRACT

Instability of flexible tubes conducting fluid, or "garden hose instability", is a phenomenon both familiar from everyday life and important for applications, which has been actively studied. However, previous works did not consider one of the most crucial physical effects — the dynamical change of the cross-section. We show how to consistently address this issue by coupling the geometrically exact rod dynamics with the fluid motion via the use of a constrained Hamilton's variational principle. We find strong effect of this dynamics on stability, and derive a variety of exact nonlinear solutions of traveling-wave type.


RÉSUMÉ

L'instabilité des tuyaux souples avec écoulement interne, ou « instabilité du tuyau d'arrosage », est un phénomène commun, étudié de longue date, et qui a d'importantes applications. Cependant, les travaux antérieurs ne tiennent pas compte d'un effet crucial : la dynamique de la section transversale du tube. Nous montrons comment l'inclure dans la dynamique en utilisant un principe de Hamilton avec contrainte, couplant la dynamique d'une tige géométriquement exacte et celle de l'écoulement interne. Nous prouvons que cela affecte l'instabilité et calculons une classe de solutions exactes de type ondes progressives.


## 1. Introduction

The phenomenon of instability of a tube carrying a fluid, also known as "garden hose instability", has been an object of active research because of its familiarity from everyday life, the fundamental questions it poses, and its practical importance. A theory of this phenomenon incorporating the dynamics of the combined motion of the tube and the fluid was developed starting from the 1950s [1,3]. The continuous theory consequently derived in [14] started a steady and active stream of studies in the area, with large portion of the contributions coming from Païdoussis and collaborators. We refer the reader to the reviews and monographs [17,20,21,8,15] for details and references. The instability theory by Païdoussis et al. combines the linearized balance of inertia, centrifugal, Coriolis, and elastic forces due to Euler's beam-like deformations in a single equation, assuming that the velocity in the tube is constant. In addition, a connection with the so-called follower-force

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approach, and some lively criticism of the latter, is available in [9]. While this theory had shown good agreement with experiment on some levels [20,11,5], there are several reasons for improvement. For example, that theory, having divergent velocities of disturbances for the short-wavelength limit, is difficult to use for the prediction of long-term dynamics in the unstable regime. In addition, while nonlinear generalizations of the model have been considered [22,21,19,13], it is difficult to consistently extend this approach to include fully nonlinear motion (twist and bend) in three dimensions. Finally, it is difficult to extend this theory to consistently take into account the dynamical change of the cross-section, as we will show below. Some of these drawbacks were addressed in the recent work [2], extending the Kirchhoff rod theory to incorporate the fluid motion inside the tube, and thus being able to predict full 3D motion in linear and weakly nonlinear regime for the case of a *constant* cross-section.

The change of the cross-section caused by the deformations of the tube is the key point of this study. Indeed, the flow of water in a real-life flexible tube can be severely limited or even completely stopped by a sharp bend in the tube. While it is possible to consider a *prescribed* changing cross-section along the tube in the generalization of Kirchhoff's approach, the description of the *dynamical* change of the cross-section has so far been elusive. Previous works have considered the effects of tube narrowing due to stretching [22,21,19,13]. However, the velocity change accommodating the variation of area was considered in the quasi-static approximation for the fluid conservation law, whereas we derive the full evolution equation for velocity. The constriction creates an extra pressure-like term, and this pressure works to undo the bend to its natural position, thus coupling deformation and fluid flow. As it turns out, this effect has an important implication for the instability and, additionally, allows for new nonlinear solutions of traveling-wave type that do not exist for a constant cross-section.

## 2. Cross-sectional dynamics in fluid conducting tubes

To describe the tube dynamics, we shall use the framework of geometrically exact rod theory [23] which is equivalent to the Kirchhoff rod theory for purely elastic rods. The configuration of the tube deforming in space is defined by: (i) the position of its line of centroids by means of the map $(s,t) \mapsto \boldsymbol{r}(s,t) \in \mathbb{R}^3$, and (ii) the orientation of cross-sections at points $\boldsymbol{r}(s,t)$. The orientation is defined by using a moving orthonormal basis $\mathbf{d}_i(s,t)$, $i=1,2,3$. One possible choice would be to assume that $\mathbf{d}_{1,2}$ are attached to the tube's cross-section (assumed to be a compact set of $\mathbb{R}^2$ with smooth boundary, usually a disk) and the third one normal to that cross-section, but not necessarily parallel to $\boldsymbol{r}_s$. However, other choices of $\mathbf{d}_i$ are possible. The physics, selected by the elastic part of the Lagrangian, selects this basis, defined up to a rigid transform in $SO(3)$ in symmetry-reduced variables. However, for the physically relevant cases, the moving basis is then described by an orthogonal transformation $\Lambda(s,t) \in SO(3)$ such that $\mathbf{d}_i(s,t) = \Lambda(s,t)\mathbf{E}_i$, where $\mathbf{E}_i$, $i=1,2,3$ is a fixed material frame. Note that $s$ is a parameter along the tube and is not necessarily the arc length. The variation of the orientation of the cross-section, i.e. the relative bend or twist, induces a change of the cross-sectional area, i.e., the area of the section of the tube perpendicular to the local centerline in the current configuration. The interior of the tube is filled with an incompressible, inviscid fluid, and we shall approximate the fluid motion by a pure one-dimensional movement from the initial position of the fluid particle $S$ to its current position at time $t$ denoted as $s=\phi(S,t)$. The fluid is thus moving along the tube with the relative velocity $u=\phi_t \circ \phi^{-1}$. The velocity of the rod in space is $\boldsymbol{v}_r = \partial_t \boldsymbol{r}$ and the velocity of fluid is $\boldsymbol{v}_f = \partial_t \boldsymbol{r} + u\partial_s \boldsymbol{r}$, as follows from time differentiating the position of the rod and a fluid particle at $s$. The physical variables describing the evolution of the rod are the local angular and linear velocities in the rod's frame, $\boldsymbol{\omega} = \Lambda^{-1}\partial_t \Lambda$ and $\boldsymbol{\gamma} = \Lambda^{-1}\partial_t \boldsymbol{r}$, and the corresponding deformations $\boldsymbol{\Omega} = \Lambda^{-1}\partial_s \Lambda$ (Darboux vector) and $\boldsymbol{\Gamma} = \Lambda^{-1}\partial_s \boldsymbol{r}$. These variables satisfy the compatibility constraints that come from equality of cross-derivatives in $s$ and $t$ as:

$$\partial_t \boldsymbol{\Omega} = \boldsymbol{\omega} \times \boldsymbol{\Omega} + \partial_s \boldsymbol{\omega}, \qquad \partial_t \boldsymbol{\Gamma} + \boldsymbol{\omega} \times \boldsymbol{\Gamma} = \partial_s \boldsymbol{\gamma} + \boldsymbol{\Omega} \times \boldsymbol{\gamma} \tag{1}$$

In order to incorporate the dynamic change of the cross-section, we assume that the cross-section $A$ is defined by a given instantaneous tube configuration, i.e. is determined by $\Lambda$, $\Lambda'$ and $\boldsymbol{r}'$. Due to the $SO(3)$ invariance of the system, we can posit a function $A = A(\boldsymbol{\Omega}, \boldsymbol{\Gamma})$ which we consider arbitrary, but given. The dependence on $\boldsymbol{\Omega}$ comes from the local frame rotating when the tube bends. The dependence on $\boldsymbol{\Gamma}$ comes from the change of cross-section due to stretching. That assumption will break down for the case of a tube with very flexible and easily stretchable walls. If $\mu$ is a typical additional pressure generated by the change of $A$ (see below), $R$ is the typical radius of the tube, $E$ is Young's modulus of the tube material, and $h$ the thickness of the wall, then the typical additional deformation is $\delta R \sim \mu R^2/Eh$. For the assumption of $A$ to depend only on the deformations to be valid, and not to be dependent on other variables, we need $\delta R \ll R$, i.e. $h \gg \mu R/E$. For example, for a rubber tube with $R = 1$ cm and $\mu \sim 1$ atm, the approximation is valid if $h \gtrsim 10^{-3}$ cm. For comparison, latex balloons and garden hoses have walls ∼10 and ∼100 times thicker.

With this physical approximation in mind and assuming that the fluid inside the tube is incompressible (in 3D), the volume conservation along the tube reads:

$$Q_t + (Qu)_s = 0, \qquad Q := A(\boldsymbol{\Omega}, \boldsymbol{\Gamma})|\boldsymbol{\Gamma}| \tag{2}$$

where the extra factor of $|\boldsymbol{\Gamma}|$ appears since $s$ is not assumed to be the arc length. If $A = A(s)$ is independent of $t$, and $|\boldsymbol{\Gamma}| = \text{const}$, (2) reduces to the conservation law $uA = \text{const}$. This is the equation for velocity used in [22,21,19,13]; however, that approach is inexact as it neglects the time variation of $A$ and stretch $\boldsymbol{\Gamma}$. In addition, setting $u \sim 1/A(\boldsymbol{\Omega})$ in the Lagrangian directly leads to badly posed problems needing an artificial regularization, even in 2D. Recall that the local