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Attainability of the Hashin–Shtrikman bounds for two-phase well-ordered composites with a nonlinear phase

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The Hashin–Shtrikman (HS) bounds for two-phase well-ordered composites are known to be attained by certain sequentially laminated constructions when the constituent phases exhibit a *linear* behavior. This implies that the bounds are optimal for that class of materials. In this Note we show that the bounds are still attained by sequentially laminated constructions when one of the phases is *nonlinear*, and that, consequently, they are optimal for a larger class of materials.

1. The effective behavior of two-phase nonlinear composites

The focus of this Note is on the viscoplastic response of two-phase composites exhibiting well-separated microstructural length scales. The instantaneous response of the constituent phases is characterized by dissipation potentials $w^{(r)}$ $(r = 1, 2)$ such that the Cauchy stress $\boldsymbol{\sigma}$ and the Eulerian strain rate $\boldsymbol{\varepsilon}$ tensors are related by:

$$\boldsymbol{\sigma} = \frac{\partial w}{\partial \boldsymbol{\varepsilon}}(\mathbf{x}, \boldsymbol{\varepsilon}), \quad w(\mathbf{x}, \boldsymbol{\varepsilon}) = \sum_{r=1}^{2} \theta^{(r)}(\mathbf{x}) w^{(r)}(\boldsymbol{\varepsilon}) \tag{1}$$

where the characteristic functions $\theta^{(r)}$ serve to describe the microstructure in the current configuration, being 1 if the position vector $\mathbf{x}$ is in phase r, and 0 otherwise. It is further assumed, as usual, that the potentials are differentiable, convex, with subquadratic growth at infinity, and such that $w^{(r)}(\boldsymbol{\varepsilon}) \geqslant 0$ and $w^{(r)}(\mathbf{0}) = 0$.

Homogenization theory states that the *effective behavior* of the composite is given by the relation between the average Cauchy stress and the average strain rate over a 'representative volume element' Ω, and that it can be characterized by an *effective dissipation potential* $\widetilde{w}$ such that—see, for instance, Refs. [1,2]:

$$\bar{\boldsymbol{\sigma}} = \frac{\partial \widetilde{w}}{\partial \bar{\boldsymbol{\varepsilon}}}(\bar{\boldsymbol{\varepsilon}}), \quad \widetilde{w}(\bar{\boldsymbol{\varepsilon}}) = \min_{\boldsymbol{\varepsilon} \in \mathcal{K}(\bar{\boldsymbol{\varepsilon}})} \langle w(\mathbf{x}, \boldsymbol{\varepsilon}) \rangle \tag{2}$$

where $\bar{\boldsymbol{\sigma}} \equiv \langle \boldsymbol{\sigma} \rangle$ and $\bar{\boldsymbol{\varepsilon}} \equiv \langle \boldsymbol{\varepsilon} \rangle$ with $\langle \cdot \rangle$ denoting volume averaging over Ω, and $\mathcal{K}(\bar{\boldsymbol{\varepsilon}})$ is the set of admissible fields $\boldsymbol{\varepsilon}$, such that there exists a continuous velocity field $\mathbf{u}$ satisfying $\boldsymbol{\varepsilon} = \nabla \otimes_s \mathbf{u}$ in Ω and $\mathbf{u} = \bar{\boldsymbol{\varepsilon}}\mathbf{x}$ on $\partial\Omega$. The optimal velocity field in (2) solves the viscoplasticity equations in Ω with a constitutive law (1).

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2. Upper bound of the Hashin–Shtrikman type

Given a set of potentials $w^{(r)}$ and of characteristic functions $\theta^{(r)}$, an upper bound for $\widetilde{w}$ can be derived by introducing a set of functions defined by:

$$v^{(r)}\left(\mathbf{L}_0^{(r)}\right) \doteq \sup_{\mathbf{E}}\left[w^{(r)}(\boldsymbol{\varepsilon}) - \frac{1}{2}\boldsymbol{\varepsilon}\cdot\mathbf{L}_0^{(r)}\boldsymbol{\varepsilon}\right], \quad r = 1, 2 \tag{3}$$

where $\mathbf{L}_0^{(r)}$ are fourth-order viscosity tensors. Then,

$$w^{(r)}(\boldsymbol{\varepsilon}) \leqslant \left[\frac{1}{2}\boldsymbol{\varepsilon}\cdot\mathbf{L}_0^{(r)}\boldsymbol{\varepsilon} + v^{(r)}\left(\mathbf{L}_0^{(r)}\right)\right] \tag{4}$$

for any $\boldsymbol{\varepsilon}$ and $\mathbf{L}_0^{(r)}$, and therefore,

$$\widetilde{w}(\bar{\boldsymbol{\varepsilon}}) \leqslant \inf_{\mathbf{L}_0^{(r)}\geqslant 0}\left[\frac{1}{2}\bar{\boldsymbol{\varepsilon}}\cdot\tilde{\mathbf{L}}_0\bar{\boldsymbol{\varepsilon}} + \sum_{r=1}^{2} c^{(r)} v^{(r)}\left(\mathbf{L}_0^{(r)}\right)\right] \tag{5}$$

where the $c^{(r)} = \langle\theta^{(r)}(\mathbf{x})\rangle$ denote the volume fractions of each phase r in Ω—such that $0 \leqslant c^{(r)} \leqslant 1$, $c^{(1)} + c^{(2)} = 1$—, and $\tilde{\mathbf{L}}_0$ is the effective viscosity tensor of a linear composite with the same microstructure $\theta^{(r)}$ as the nonlinear composite, but with a local linear behavior characterized by viscosity tensors $\mathbf{L}_0^{(r)}$. Various derivations of this bound have been given by Ponte Castañeda [3,4], Willis [5], Talbot & Willis [6], and Idiart & Ponte Castañeda [7], among others.

The first term in (5) can be bounded from above by the linear Hashin–Shtrikman bounds of Willis [8]. These bounds are a generalization of the original bounds of Hashin & Shtrikman [9], and can be written in terms of an effective viscosity tensor:

$$\tilde{\mathbf{L}}_{\mathrm{HS}} = \left(\sum_{r=1}^{2} c^{(r)}\left[\mathbf{I} + \left(\mathbf{L}_0^{(r)} - \mathbf{L}_0\right)\mathbf{P}_0\right]\right)^{-1}\left(\sum_{r=1}^{2} c^{(r)}\left[\mathbf{I} + \left(\mathbf{L}_0^{(r)} - \mathbf{L}_0\right)\mathbf{P}_0\right]^{-1}\mathbf{L}_0^{(r)}\right) \tag{6}$$

where

$$\mathbf{P}_0 = \int_{|\mathbf{n}|=1} \mathbf{H}_0(\mathbf{n})\nu(\mathbf{n})\,\mathrm{d}\mathbf{n}, \qquad H_{0_{ijkl}} = K_{ik}^{-1}(\mathbf{n}) n_j n_l|_{(ij)(kl)}, \quad K_{ik} = L_{0_{ijkl}} n_j n_l \tag{7}$$

is a microstructural tensor that depends on the H-measure $\nu(\mathbf{n})$ of the composite microstructure. H-measures are geometrical objects—introduced as such by Tartar [10] and Gérard [11]—that depend on the two-point microstructural correlations $\langle\theta^{(r)}(\mathbf{x})\theta^{(s)}(\mathbf{x}+\mathbf{z})\rangle$; they quantify in the phase space the lack of compactness of weakly converging sequences of characteristic functions $[\theta^{(r)}(\mathbf{x}) - c^{(r)}]$—see Ref. [12]—and provide a partial characterization of the microstructural oscillations along different directions in the physical space. Explicit expressions of $\nu(\mathbf{n})$ for different types of microstructures can be found, for instance, in [8,12,13]. In any event, the effective viscosity tensor (6) is such that:

$$\tilde{\mathbf{L}}_0 \leqslant \tilde{\mathbf{L}}_{\mathrm{HS}} \quad \text{for any } \mathbf{L}_0 \text{ satisfying} \quad \mathbf{L}_0^{(r)} \leqslant \mathbf{L}_0 \quad \text{for all } r \tag{8}$$

inequalities that can be interpreted in the sense of quadratic forms. Introducing (6) in (5) leads to a nonlinear upper bound of the Hashin–Shtrikman type of the form [6,7]:

$$\widetilde{w}(\bar{\boldsymbol{\varepsilon}}) \leqslant \widetilde{w}_{\mathrm{HS}}(\bar{\boldsymbol{\varepsilon}}) \doteq \inf_{\mathbf{L}_0^{(r)}\geqslant 0}\left[\frac{1}{2}\bar{\boldsymbol{\varepsilon}}\cdot\tilde{\mathbf{L}}_{\mathrm{HS}}\bar{\boldsymbol{\varepsilon}} + \sum_{r=1}^{2} c^{(r)} v^{(r)}\left(\mathbf{L}_0^{(r)}\right)\right] \tag{9}$$

for any $\mathbf{L}_0$ satisfying (8). The function $\widetilde{w}_{\mathrm{HS}}(\bar{\boldsymbol{\varepsilon}})$ bounds from above the effective potential of any composite within the class $\mathcal{C}(w^{(r)}, c^{(r)}, \nu)$ of two-phase composites with prescribed local potentials $w^{(r)}$, volume fractions $c^{(r)}$, and H-measure $\nu(\mathbf{n})$.

3. A class of two-phase nonlinear composites with an attainable upper bound

When the local responses are *linear* and well-ordered, the bound $\widetilde{w}_{\mathrm{HS}}$ is known to be attained for all possible values of $c^{(r)}$ and $\nu(\mathbf{n})$ by certain sequentially laminated constructions—see, for instance, Refs. [1,14] and references therein. The attainability of the bound when one of the phases is *nonlinear* has been demonstrated by Ponte Castañeda [4,15] for isotropic potentials $w^{(r)}$ that depend on the first and second invariants of $\boldsymbol{\varepsilon}$. In this section, we show that the bound $\widetilde{w}_{\mathrm{HS}}$ is attainable for the more general class $\mathcal{S}(w^{(r)}, c^{(r)}, \nu) \subset \mathcal{C}(w^{(r)}, c^{(r)}, \nu)$ of two-phase composites with potentials $w^{(r)}$ such that: