



Mixed boundary value problem in Potential Theory: Application to the hydrodynamic impact (Wagner) problem

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ABSTRACT

A three-dimensional solution of the mixed boundary value problem posed in Potential Theory is proposed. The support of the Neumann condition is conformally mapped onto a unit disk. On that disk, the solution is broken down as Fourier series of azimuthal angle and linear combinations of known functions of the radial coordinate. It is shown that the whole problem reduces highly nonlinear equations for the coefficients of the mapping function. The present method of solution is to be applied to hydrodynamic impact problem.

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1. Introduction

A mixed Neumann–Dirichlet boundary value problem is posed in Potential Theory. The Laplace equation is solved in a half three-dimensional space. This configuration is relevant in several domains of the physics; among them a charged disk in electrostatics, or the hydrodynamic impact problem known as the linearized Wagner problem [1]. This latter problem is even more complicated since the line which bounds the two supports of the two boundary conditions is a part of the solution.

We propose here a new mathematical solution of that Wagner problem. A semi-analytical solution of the problem is hence established as a series of known functions whose properties are checked and analyzed. The proposed method has not been studied for practical cases yet. However some physical properties of the solution, like hydrodynamic loads are formulated. It is also expected that the present solution will help to propose new criteria regarding the stability of the solution in connection with the regularity of the shape at the initial contact point.

2. Method of solution

An illustration of the linearized three-dimensional Wagner problem is shown in Fig. 1. A Neumann condition is prescribed on a closed area D . A homogeneous Dirichlet condition is prescribed on the complementary surface F . Following Korobkin [2] and Howison et al. [3] among others, the corresponding boundary value problem is formulated in terms of the displacement potential ϕ as follows:

$$\begin{cases} \phi_{,xx} + \phi_{,yy} + \phi_{,zz} = 0, & z < 0 \\ \phi = 0, & z = 0, (x, y) \in F \\ \phi_{,z} = f^*(x, y, t), & z = 0, (x, y) \in D \\ \phi \rightarrow 0, & (x^2 + y^2 + z^2) \rightarrow \infty \end{cases} \quad (1)$$

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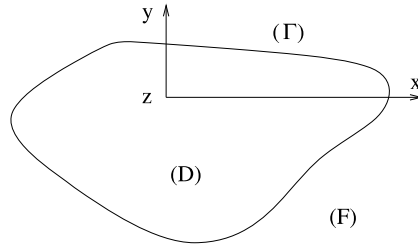


Fig. 1. Illustration of the linearized three-dimensional Wagner problem.

where f^* is a regular enough function of the Cartesian coordinates (x, y) and time t . By using a theorem by Zaremba [4] applied to the vertical displacement $V = -\phi_{,z}$, it is shown that the field V is induced by a unique distribution of sources over the surface D . The source intensities identify with the planar Laplacian $\Delta_2 \phi = \phi_{,xx} + \phi_{,yy} = S(x, y, t)$. In addition $V(x, y, z, t)$ is continuous throughout the fluid domain $z \leq 0$ including its boundary $z = 0$ and the contact line Γ . After some algebra of the integral representation of field V , the following integral equation is obtained (see Korobkin [2]):

$$\frac{1}{2\pi} \int_D \frac{S(x_0, y_0, t) dx_0 dy_0}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = f^*(x, y, t) \quad (2)$$

Additional conditions are prescribed along the contact line. This condition implies that not only the displacement potential $\phi(x, y, 0, t)$ and the vertical displacement $\phi_{,z}(x, y, 0, t)$ but also the horizontal displacements $\phi_{,x}(x, y, 0, t)$ and $\phi_{,y}(x, y, 0, t)$ are continuous through Γ . Those conditions are necessary in order to determine the contact line Γ which is unknown.

By using the Riemann mapping theorem (see Nehari [5, pp. 173–174]), we introduce the conformal mapping function g which turns the surface D onto a unit disk C_1

$$Z = x + iy = g(\omega), \quad \omega = \xi + i\eta \quad (3)$$

where Z denotes a complex number in the physical plane $z = 0$. As function g is conformal (hence analytical), its derivative never vanishes and the following formula is helpful to express the Green function of the problem.

$$\frac{1}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = \frac{1}{|g(\omega) - g(\omega_0)|} = \frac{1}{|g'(\omega)|} \frac{1}{|\omega - \omega_0|} + \frac{T(\omega, \omega_0)}{|\omega - \omega_0|} \quad (4)$$

The function $T(\omega, \omega_0)$ is regular since the leading order term behaves as $O(|\omega - \omega_0|)$. By substituting (4) in the integral equation (2), it can be shown that S is a solution of

$$\frac{1}{2\pi} \int_{C_1} \frac{S(\omega_0) d\sigma_0}{|\omega - \omega_0|} = Q(\rho, \alpha, t), \quad \phi = 0, \quad \text{and} \quad \phi_{,n} = 0, \quad \text{at} \quad |\omega| = 1 \quad (5)$$

where the right-hand side Q depends on the solution S , but it is regular enough all over D . The integral equation in combination with boundary conditions (5) proves that ϕ is the solution of the following boundary value problem now posed on the unit disk

$$\begin{cases} \phi_{,\rho^2} + \frac{1}{\rho} \phi_{,\rho} + \phi_{,\tilde{z}^2} + \frac{1}{\rho^2} \phi_{,\alpha^2} = 0, & \tilde{z} < 0 \\ \phi = 0, & \tilde{z} = 0, \rho > 1 \\ \phi_{,\tilde{z}} = Q(\rho, \alpha, t), & \tilde{z} = 0, 0 < \rho < 1 \\ \phi \rightarrow 0, & (\tilde{z}^2 + \rho^2 \rightarrow \infty) \end{cases} \quad (6)$$

where the vertical coordinate \tilde{z} is formal, its connection with the original variable z is complicated and is not discussed here. The method for solving the problem (6) is described by Korobkin and Scolan in [6]. Provided that Q has a Fourier transform of the azimuthal variable α and a polynomial expansion with ρ , say

$$Q(\rho, \alpha, t) = \Re \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} q_k^{(n)} \rho^k e^{in\alpha}, \quad \rho \in [0:1] \quad (7)$$

after some algebra, it is shown that the displacement potential ϕ reads

$$\phi(\rho, \alpha, 0, t) = \frac{2}{\pi} \Re \sum_{n=0}^{\infty} \rho^n \sum_{k=0}^{\infty} q_k^{(n)} z_{k+n} [\sqrt{1-\rho^2} + (k-n)D_{k-n}(\rho)] e^{in\alpha} \quad (8)$$

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