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Averaging in variational inequalities with nonlinear restrictions along manifolds ☆

Homogénéisation de inégalités variationnelles avec restrictions non linéaires sur une variété

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ABSTRACT

We consider variational inequalities for the Laplace operator in a domain Ω of $\mathbb{R}^n$ periodically perforated along a manifold, with nonlinear restrictions for the flux on the boundary of the cavities. We assume that the perforations are balls of radius $O(\varepsilon^\alpha)$ distributed along a $(n-1)$-dimensional manifold γ with period ε. Here $\varepsilon > 0$ is a small parameter, $\alpha > 0$ and $n \geqslant 3$. On the boundary of the perforations, we have the restrictions for the solution $u_\varepsilon \geqslant 0$, $\partial_\nu u_\varepsilon \geqslant -\varepsilon^{-\kappa}\sigma(x, u_\varepsilon)$ and $u_\varepsilon(\partial_\nu u_\varepsilon + \varepsilon^{-\kappa}\sigma(x, u_\varepsilon)) = 0$, where $\kappa \geqslant 0$ and σ is a certain smooth function. For $\alpha \geqslant 1$ and $\kappa = (\alpha-1)(n-2)$, we characterize the asymptotic behavior of u_ε as $\varepsilon \to 0$ providing the homogenized problems. A critical size of the cavities is found when $\alpha = \kappa = (n-1)/(n-2)$ for which the corrector in the energy norm is constructed.

RÉSUMÉ

Nous considèrons inégalités variationnelles pour l'opérateur de Laplace dans une domaine Ω de $\mathbb{R}^n$ périodiquement perforé, et avec des restrictions pour le flux sur la frontière des trous. On suppose que les perforations sont des boules de rayon $O(\varepsilon^\alpha)$ distribuées sur une variété de dimension $(n-1)$, γ, de période ε. Ici $\varepsilon > 0$ est une petite paramètre, $\alpha > 0$ et $n \geqslant 3$. Sur la frontière des trous nous avons des restrictions pour la solution $u_\varepsilon \geqslant 0$, $\partial_\nu u_\varepsilon \geqslant -\varepsilon^{-\kappa}\sigma(x, u_\varepsilon)$ et $u_\varepsilon(\partial_\nu u_\varepsilon + \varepsilon^{-\kappa}\sigma(x, u_\varepsilon)) = 0$, où $\kappa \geqslant 0$ et σ est une certaine fonction régulière. Pour $\alpha \geqslant 1$ and $\kappa = (\alpha-1)(n-2)$, nous caractérisons le comportement asymptotique de u_ε pour $\varepsilon \to 0$. On trouve les problèmes homogéneisés et une taille critique des trous pour $\alpha = \kappa = (n-1)/(n-2)$. Pour cette taille on construit le correcteur dans la norme de l'énergie.

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1. Introduction

In this Note, we consider the solution u_ε of a variational inequality for the Laplace operator in a domain Ω_ε perforated along a $(n-1)$-dimensional manifold with a nonlinear adsorption rate on the boundary S_ε of the cavities G_ε. Ω_ε denotes the perforated domain $\Omega \setminus G_\varepsilon$, Ω a domain of $\mathbb{R}^n$ with $n \geqslant 3$, and the nonlinear term involves a large parameter and a continuously differentiable function $\sigma = \sigma(x,u)$ defined in $\bar{\Omega} \times \mathbb{R}$, which is strictly monotonic with respect to u. We assume that the perforations G_ε are the unions of balls of radius $C_0\varepsilon^\alpha$ with $C_0 > 0$ and α ranges in $[1,\infty)$. These perforations are periodically distributed along the manifold $\gamma = \Omega \cap \{x_1 = 0\} \neq \emptyset$ with period ε. Here, $\varepsilon > 0$ denotes a parameter that we shall make converge towards zero. On the boundary of the cavities S_ε (the union of the boundaries of the balls), we consider the nonlinear restrictions (5) involving the parameter $\varepsilon^{-\kappa}$ with $\kappa = (\alpha-1)(n-1)$. We study the asymptotic behavior of the solution u_ε as $\varepsilon \to 0$ of the problem, namely of problem (3)–(4) for a given data $f \in L^2(\Omega)$. Note that, among the possible values of α and κ, here we consider those such that the parameter $\varepsilon^{-\kappa}$ multiplied by the area of S_ε is of order $O(1)$. We also emphasize that the restrictions on S_ε are different from those considered in previous homogenization problems in the literature of applied mathematics. The problem arises in the framework of the modeling of the diffusion of substances in porous media: see [1] and [2] for more precise models.

For $\alpha \in [1,(n-1)/(n-2)]$, we obtain the weak convergence of the solution u_ε, when $\varepsilon \to 0$, as stated in Theorems 3.1 and 3.3, to the solution u of a problem for the Laplace operator in Ω with a certain *homogenized transmission condition* on γ. This transmission condition contains a nonlinear function of u which represents the macroscopic contribution of the nonlinear law on the boundary of the microscopic cavities. The nonlinear term is obtained from the function σ depending on the value of α in (1): see (14) for $\alpha \in [1,(n-1)/(n-2))$ and (7) for $\alpha = (n-1)/(n-2)$. Note that the case where $\alpha = (n-1)/(n-2)$ differs from the rest of the cases since we obtain a boundary value problem and the nonlinear term is different: it involves a new function $H(x,u)$ defined implicitly by the nonlinear equation (8), which proves to have similar properties to the given function σ (cf. (2)). This value for parameters α and κ, namely $\alpha = \kappa$ in (1), provides a *critical size* of the balls G_ε. See Remark 1 in this respect. In the case where $\alpha > (n-1)/(n-2)$ the homogenized problem is the Dirichlet problem (15).

Similar geometrical configurations for linear and nonlinear boundary value problems have been considered in many previous papers: let us mention [1–9] for some of these problems and for further references. Also let us mention [10,11] for the homogenization of variational inequalities. We refer to [3] as the closest problem to the problem here considered. In [3] a nonlinear boundary condition on S_ε has been considered, namely $\partial_\nu u_\varepsilon + \varepsilon^{-\kappa}\sigma(x,u_\varepsilon) = 0$ for the value $\kappa = \alpha$. [1] considers the same boundary condition but with the cavities periodically distributed on the whole volume and with $n = 3$. In this connection, let us mention [4] for non-homogeneous boundary conditions, [2] and [5] for $\alpha = 1$, and [6] for evolution problems.

It is worth mentioning that for different homogenization problems, with different homogenized equations in Ω and on γ, the kind of nonlinear equation (8) also appears in [1] and [3] respectively. The change of type of nonlinearity was first noticed in [1] for spatially distributed cavities and in [3] for the cavities along γ. This recalls the so-called *strange term* arising in many papers on homogenization problems with critical sizes: see, e.g., [8] for different linear problems and further references, and [1] and [3] for nonlinear boundary value problems. In the present paper, we highlight the phenomena for problems with strong nonlinear restrictions on the boundary of the microscopic cavities.

It should be noted that, since we are dealing here with homogenization of variational inequalities, and nonlinear restrictions on the boundary of the perforations, proofs rely on extension operators, on transformations of surface integral on S_ε into volume integrals in Ω_ε, on convergence of measures, and on the appropriate choice of positive test functions (cf. (12) and Remark 1) which allows us to pass to the limit in the weak formulations. The main convergence results are stated in Theorems 3.1, 3.3 and 3.4. Furthermore, an improved approximation for the macroscopic solution is constructed when $\alpha = \kappa$, and more accurate results are obtained with respect to the energy norm (cf. Theorem 3.2 and [12] for other values of α and κ). For the sake of brevity, we only provide a sketch of the proofs involving the critical size, leaving the technical and laborious computations, and the rest of the proofs, to be performed in a forthcoming publication (cf. [12]). Finally, the structure of the paper is as follows: Section 2 contains the setting of the ε-dependent problem while Section 3 contains the homogenized problems and the corrector result.

2. Setting of the ε-dependent problem

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geqslant 3$, with a smooth boundary $\partial\Omega$. Assume that $\gamma = \Omega \cap \{x_1 = 0\} \neq \emptyset$ is a domain on the hyperplane $\{x_1 = 0\}$. We denote by G_0 the ball of radius 1 centered at the origin of coordinates. For a set B, and $\delta > 0$, we denote by $\delta B = \{x \mid \delta^{-1}x \in B\}$. We set

$$\tilde{G}_\varepsilon = \bigcup_{z' \in \mathbb{Z}'} \left(a_\varepsilon G_0 + \varepsilon z'\right) = \bigcup_{j \in \mathbb{Z}'} G_\varepsilon^j$$

where $\mathbb{Z}'$ is the set of vectors of the form $z' = (0, z_2, \ldots, z_n)$ with integer components z_l, $l = 2, \ldots, n$, $a_\varepsilon = C_0\varepsilon^\alpha$, C_0 is a positive number, ε is a small positive parameter that we shall make converge towards zero, and α is a parameter, $\alpha \geqslant 1$. If no confusion arises, we identify $z \in \mathbb{Z}'$ with $j \in \mathbb{Z}'$, and we define