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Dissipative propagation of pressure waves along the slip-lines of yielding material

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ABSTRACT

Traditional slip-line field theory considers steady-state pressure distributions. This work examines the *evolution* of pressure at yield. We show that, for a two phase material in plane strain at the point of plastic yield of the main constituent, pressure develops according to dissipative wave mechanics. The pressure wave equations are examined in more detail for the special case of radially symmetric plane strain, and for Von Mises and Drucker-Prager yield envelopes. The positions of pressure troughs are compared to the separation of slip-lines formed during the expansion by internal pressurisation of a cylinder of mild steel.

1. Introduction

This work examines the evolution of pressure of plastically yielding material. It will show that, for a two phase material in plane strain, pressure propagates along slip-lines in dissipative waves.

Slip-lines are the characteristics of the quasistatic momentum balance and are used in the design or description of, among other things, welds (Khan et al., 2014), hardness tests (Jackson, Ghaednia, & Pope, 2015), retaining walls (Vo & Russell, 2014) and pipeline foundations (Gao, Wang, & Zhao, 2015). A particularly elegant application of slip-line field theory (SFT) for Von Mises and Drucker-Prager materials, and one relevant to the plastic design of pressure vessels (Sowerby & Johnson, 1970), arises for radial stress of a cylinder in plane-strain. As Hill (1950) and Shield (1953) discussed, slip-lines are both the only curves across which pressure and velocity fields can have discontinuities, and are directed at a constant angle to the direction of maximal tensile stress. Thus SFT predicts that any strain localisations will occur along logarithmic spirals. Sowerby and Johnson (1970) and Papamichos, Vardoulakis, Tronvoll, and Skjaerstein (2001) verified these predictions via pressurisation experiments from which come beautiful networks of intersecting failure lines following logarithmic spirals. Figure 9.3a of Sowerby and Johnson (1970) appears here as Fig. 4a.

Thus the quasistatic momentum balance $\nabla \cdot \boldsymbol{\sigma} = \mathbf{0}$ determines the geometry of lines of failure for a steel cylinder internally pressurised to yield. However, the analysis does not identify the spacing of failure lines which, at least in Fig. 4a,

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List of symbols

Latin symbol	Description
a/b	Constant coefficients used in separation of variables.
$\boldsymbol{A}/\boldsymbol{B}$	The matrices describing the part of the rate of change of momentum which is proportional to the derivative of $(p, \theta)^T$ with respect to $x_{1/2}$.
B	The Biot coefficient.
c	The cohesion of a Drucker-Prager material.
C	The drag coefficient.
$c_n/d_n/f_n$	Fourier coefficients.
c_s	The speed of second sound.
c_u	The wave speed under undrained conditions.
d	Half the distance separating wide parallel plates. (This symbol appears only in the appendices.)
d	The negative of the constant of proportionality between the β-volume fraction ϕ and specific interaction force $\boldsymbol{f}$. (This symbol appears only in a footnote.)
$\frac{\mathrm{D}^{(\alpha/\beta)}}{\mathrm{D}t}$	The α/β material derivative.
$\boldsymbol{f}$	The interaction force (exerted by β on α).
$g(p)$	The shear stress at which yield occurs at hydrostatic pressure p.
G_α	The shear modulus of α. (This symbol appears only in the appendices.)
H	A signature time scale.
k	The shear strength of a Von Mises material.
K	The bulk modulus relating pressure and volumetric strain.
K_s	The bulk modulus of the skeleton.
$K_{\alpha/\beta}$	The bulk modulus of α/β. (This symbol appears only in the appendices.)
l	The distance of steady-state laminar flow of an incompressible Newtonian fluid. (This symbol appears only in the appendices.)
L	A geometric length scale; the total length of a slip-line.
n	A positive integer.
n^*	An upper/lower bound of the range of n corresponding to hyperbolic/trigonometric pressure eigenmodes p_n.
$n_{\max}$	The index of the highest pressure eigenmode used to produce plots.
p	The hydrostatic component of the effective stress.
P	The pressure on an incompressible Newtonian fluid. (This symbol appears only in the appendices.)
p_D	The discontinuous part of the pressure.
p_f	The pore fluid pressure. (This symbol appears only in a footnote.)
p_o	The constant total pressure at the outer wall of the cylinder.
p_Q	The steady state solution to the pressure for vanishing specific discharge, $\boldsymbol{\mathcal{V}}$.
$\hat{p}$	The overpressure, $p - p_Q$.
$\tilde{p}$	The dimensionless overpressure.
$\check{p}$	The rescaled dimensionless overpressure.
$\overset{\sqcap}{p}$	The continuous part of the overpressure.
$\left\lvert\hat{p}\right\rvert_{\max}\Big/\left\lvert\overset{\sqcap}{p}\right\rvert_{\max}$	Attenuation factors used to plot overpressures/the continuous parts of overpressures.
q	The shear component of the effective stress.
r	The radial polar coordinate.
r_0	The value of r where $s_1 = s_{10}$.
r_i	The value of the radial coordinate at the inner wall of the cylinder.
r_o	The value of the radial coordinate at the outer wall of the cylinder.
S	A function of $\tilde{s}_1$ used in separation of variables.
s_{10}	The value of s_1 where $r = r_0$.
s_i	Coordinates the level sets of which are characteristics of the momentum balance.
$\tilde{s}_i$	The dimensionless coordinates.
T	A function of $\tilde{t}$ used in separation of variables.
$\tilde{t}$	The dimensionless time.
$\boldsymbol{u}$	The displacement.
$\bar{v}$	The mean speed of the incompressible Newtonian fluid. (This symbol appears only in the appendices.)
$\boldsymbol{v}^{(1/2)}$	The velocity of α/β.
$\boldsymbol{\mathcal{V}}$	The specific discharge of β.