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Exact solution of Eringen's nonlocal integral model for bending of Euler–Bernoulli and Timoshenko beams

Meral Tuna, Mesut Kirca*

Istanbul Technical University, Faculty of Mechanical Engineering, Inonu Cad. No:65 34437, Gumussuyu, Beyoglu, Istanbul, Turkey

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ABSTRACT

Despite its popularity, differential form of Eringen nonlocal model leads to some inconsistencies that have been demonstrated recently for the cantilever beams by showing the differences between the integral and differential forms of the nonlocal equation, which indicates the importance and necessity of using the original integral model.

With this motivation, this paper aims to derive the closed-form analytical solutions of original integral model for static bending of Euler Bernoulli and Timoshenko beams, in a simple manner, for different loading and boundary conditions. For this purpose, the Fredholm type integral governing equations are transformed to Volterra integral equations of the second kind, and Laplace transformation is applied to the corresponding equations.

The analytical expressions of the beam deflections which are obtained through the utilization of the proposed solution technique are validated against to those of other studies existing in literature. Furthermore, for all boundary and loading conditions, in contrast to the differential form, it is clearly established that the integral model predicts the softening effect of the nonlocal parameter as expected. In case of Timoshenko beam theory, an additional term that includes the nonlocal parameter is introduced. This extra term is related to the shear rigidity of the beam indicating that the nonlocal effect manifests itself via not only bending, but also shear deformation.

1. Introduction

Local theory of elasticity becomes inadequate when internal (e.g. atomic or granular distance, relaxation time etc.) and external (e.g. crack length, wave length, period of load, application area of load etc.) characteristic length or time scales are comparable as appeared in several cases such as sharp crack tip propagation in fracture mechanics, wave propagation of composites under high-frequency excitations and mechanical behavior of nano and micro structures (Benvenuti & Simone, 2013; Eringen, 1974; Fernández-Sáez, Zaera, Loya, & Reddy, 2016; Pisano & Fuschi, 2003; etc.) where the nonlocal effects are much more dominant.

In order to overcome this shortcoming, as well as to investigate the size effects, different theories have been developed. First attempts to address the nonlocal effects stand back to works of Cauchy and Voight (19th century) and Cosserat (20th century) (Fernández-Sáez et al., 2016). The gradient elasticity constitutive models were developed by Mindlin (1965), Toupin (1962) and Mindlin and Eshel (1968). Meanwhile, early formulations of nonlocal elastic constitutive equations which

* Corresponding author.
E-mail addresses: tunamer@itu.edu.tr (M. Tuna), kircam@itu.edu.tr (M. Kirca).

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were introduced by Kröner (1967), Krumhansl (1968) and Kunin (1968) were improved further by Eringen (1966, 1972, 1983, 1987), Eringen and Edelen (1972). In addition to abovementioned approaches, molecular and atomistic theories (e.g. molecular dynamic simulations) have also been offered as effective methods, although huge amount of computational effort should be allocated for large systems (Ansari, Rouhi, & Mirnezdah, 2014; Nazemizadeh & Bakhtari-Nejad, 2015; etc.). Despite their significance, no further information about the advantages, capabilities, and applications of each theory will be given here since it is not the scope of the study. Some examples of other considerable works on the topics can be found in the articles of Benvenuti and Simone (2013), Pisano, Sofi, and Fuschi (2009), Li, Yao, Chen, and Li (2015), Eltaher, Khater, and Emam (2015) and Fernández-Sáez et al. (2016), etc.

In the literature, one of the most widely used methods is Eringen's nonlocal theory of linear elasticity which incorporates an internal length parameter into the constitutive equation to capture the microstructural effects. Theory exhibits a convolution format for constitutive relation where stress at each point is related to the strain of entire domain, through a kernel function that is inversely proportional to the distance between investigated and neighboring points. In addition to classical integral model (also known as fully nonlocal model), an alternative expression is proposed, which is known as two-phase local/nonlocal model, where both local and nonlocal integral constitutive equations are included through a fraction coefficient that regulate their weights (Eringen, 2002). Furthermore, the spatial integrals encountered in the formulations of nonlocal theory can be converted to their equivalent differential form for specific types of kernel functions as indicated by Eringen (1983).

Due to its simplicity, many studies utilize the differential form of Eringen model in order to investigate the bending, buckling, vibration and wave propogation behavior of structural elements such as; rods, tubes, beams, plates and shells (Anjomshoa, Shahidi, Hassani, & Jomehzadeh, 2014; Dansehmehr, Rajabpoor, & Hadi, 2015; Fotouhi, Firouz-Abadi, & Haddadpour, 2013; Hosseini-Hashemi et al., 2013; Hu, Liew, Wang, He, & Yakobson, 2008; Jalali, Jomehzadeh, & Pugno, 2016; Lu et al., 2007; Nejad Hadi & Rastgo, 2016; Phadikar & Pradhan, 2010; Rahmani & Pedram, 2014; Reddy, 2007; Reddy & Pang, 2008; Roque, Ferreira, & Reddy, 2011; Salehipour , Shahidi & Nahvi, 2015; Shaat, 2015; Wang & Liew, 2007; etc.). Before any further progress, it should be mentioned that; although a few pioneering studies are referenced here, there is an extensive literature about the field. In this regard, the readers are encouraged to check the articles of Arash and Wang (2012), Eltaher et al. (2015), Khodabakshi and Reddy (2015) and Fernández-Sáez et al. (2016) to access more publications.

Despite the popularity of the differential form of Eringen model, in several studies focusing on the bending behavior of cantilevered beams insubstantial results are presented (Hu et al., 2008; Peddieson, Buchanan, & McNitt, 2003; Reddy & Pang, 2008; Challamel & Wang, 2008; Li et al., 2015; etc.). In this regard, it is reported that the cantilever beams subjected to concentrated forces are insensitive to nonlocal (small-scale) parameters while in the case of uniformly distributed load nonlocal effect manifests itself as a stiffening contribution which is a questionable outcome since a softening behavior is expected within the scope of the results obtained for other boundary conditions (i.e., Reddy & Pang, 2008). To overcome this deficiency, Challamel and Wang (2008) propose a new model that couples integral model and gradient model which is based on the combination of the local and nonlocal curvatures in the constitutive equation. Furthermore, Benvenuti and Simone (2013) point out that fully nonlocal model is unable to capture the nonlocal effects of a rod (e.g. constant strain field under constant tensile stress and inconsistent stress-strain relations in the case of distributed axial load). They recover the size effects by converting the two-phase integral formulation into a specific gradient elasticity formulation.

In addition, there are also some other studies that do not utilize from the differential counterpart of Eringen integral equations. For instance, Polizzotto (2001) developes the Eringen nonlocal integral model by assuming an attenuation function depending on a geodetic distance, and accordingly derives the variational statements to obtain a nonlocal finite element formulation. In another study, Pisano and Fuschi (2003) convert the two-phase integral formulation into Volterra integral equations by utilizing from the symmetry property of a specific kernel function to examine the behavior of a bar under tension. Following the conversion, the exact solution of Volterra type integral equations are obtained by using the method of successive approximations by Neumann's series. Furthermore, Khodabakhshi and Reddy (2015) provide a general finite element formulation for the local/nonlocal two-phase integral equations and investigate the behavior of Euler–Bernoulli beams under transverse loads. Although the deflection of a simply supported beam is not in good agreement with literature, the aforementioned inconsistency encountered for cantilever beams is suppressed. In their recent study, Fernández-Sáez et al. (2016) indicated that the solution of Eringen integral equation coincidences with the differential form of Eringen model if corresponding boundary conditions (see; Polyanin & Manzhirov, 2008) are satisfied, which is highlighted earlier by Benvenuti and Simone (2013). In the light of this information, Fernández-Sáez et al. (2016) propose a general method to solve the integral equation, and correct the paradoxical behavior encountered with cantilever beams. Results are compared with widely used differential Eringen model and it is concluded that differential form is unable to capture the nonlocal effects correctly.

The present work is motivated by the fact that closed-form exact solution of Eringen integral model has not been developed so far, despite the inconsistent results obtained from differential form, so called counterpart of integral one. Therefore, the aim of this study is to derive the analytical expression of solution of Eringen integral model. For this purpose, the general three-dimensional equations are reduced to one-dimensional form to formulate the Euler Bernoulli and Timoshenko beams. Fredholm type integral equations are split into three parts that includes two Volterra integral equations of the second kind. Although the proposed method is valid for any kernel function that depends on the distance variable, it is taken similar to those of Fernández-Sáez et al. (2016), Pisano and Fuschi (2003), Reddy (2008), in order to make comparisons. The solution of integral equation is obtained by using Laplace transformation (Wazwaz, 2011). The non-dimensional analytical