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Incremental constitutive models for elastoplastic materials undergoing finite deformations by using a four-dimensional formalism

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ABSTRACT

When constructing incremental constitutive models of elastoplasticity for materials undergoing finite deformations, the tensors and their rates should respect the principle of frame-indifference. Instead of classical 3D approaches in which different objective transports may be arbitrarily used in the constitutive equations, we propose to model the constitutive equations using the four-dimensional formalism of the theory of Relativity. This formalism ensures that any 4D tensor is frame-indifferent thanks to the principle of covariance. It is further possible to define 4D rate operators that are all, by construction, frame-indifferent. Among these covariant rates, the 4D Lie derivative is chosen to construct incremental constitutive relations because it is invariant to the superposition of rigid body motion. A 4D rate type model of elastoplasticity with isotropic hardening is thus developed and compared with existing classical 3D constitutive models of elastoplasticity established in the context of finite deformations.

1. Introduction

Over this past century, tremendous progresses have been achieved concerning the mathematical aspects of plasticity, especially in the framework of infinitesimal deformations (Banks, Hu, & Kenz, 2011; Borja, 2013; Chaboche, 2008; Osakada, 2010). However, in the context of finite deformations (Bertram, 2012; Hill, 1983; Lubarda, 2002; Nemat-Nasser, 2004), open issues still exist concerning, in particular, the subject of material objectivity (Frewer, 2009; Shutov & Ihlemann, 2014; Truesdell & Noll, 2003). Because plastic models are usually constructed with incremental variables, a frame-indifferent rate has to be chosen (S. Khan & Huang, 1995). Even though different kinds of objective transports can be defined in classical 3D mechanics (Green & Naghdi, 1965; Nemat-Nasser, 2004; Oldroyd, 1950; Shutov & Ihlemann, 2014; Truesdell, 1955), it is difficult, *a priori*, to select one transport versus another on the basis of physical or mathematical considerations (S. Khan & Huang, 1995).

To consistently deal with material objectivity issues in continuum mechanics, the principle of covariance in four dimensions used usually in the theory of Relativity offers an interesting framework. Covariance corresponds to the fact that 4D

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tensors and equations are inherently indifferent to change of observers. This formalism also offers the possibility to distinguish the principle of frame-indifference from the invariance with respect to the superposition of rigid body motion, whereas both correspond to the same notion in a classical 3D context. The 4D formalism further enables to define properly a time derivative that is invariant to the superposition of rigid body motion. This derivative is the Lie derivative (Rouhaud, Panicaud, & Kerner, 2013). It is thus a relevant formalism to construct constitutive models for elastoplasticity in finite deformations (Frewer, 2009; Panicaud & Rouhaud, 2014; Rouhaud et al., 2013).

Unlike classical continuum mechanics, which is constructed in the 3D space, we thus propose to describe the finite deformations of a material body using the 4D formalism of the theory of Relativity. It is important to precise that the scope of the present work concerns materials undergoing deformations for which the velocity is very small compared to the velocity of light. The objective of this work is thus to describe materials that are classically studied in continuum mechanics, but using the tools of 4D physics. The framework of the theory of Newton is thus replaced by the theory of Relativity to describe the same classical material continuum. This paper first presents a brief introduction to the 4D formalism used in the theory of Relativity and its application to plasticity. The construction of 4D models for elastoplasticity is next proposed. The projection of the 4D models on a 3D space leads to new 3D constitutive models. As an illustration, numerical simulations are performed for simple loading cases and compared with results from existing 3D models.

2. 4D space-time framework

2.1. 4D coordinate systems and frames of reference

In the theory of Relativity, space-time is described with a four-dimensional differentiable manifold (Boratav & Kerner, 1991). A set of four coordinates denoted:

$$\xi^{\mu} = (\xi^1, \xi^2, \xi^3, \xi^4) = (\xi^i, ct) \tag{1}$$

is used to parameterize a point of this manifold and this set constitutes an event. The index notation is used and Greek indices (μ, ν ... running from 1 to 4) label the 4D space-time entities, while the Roman indices (i, j ... running from 1 to 3) label the spatial part of the entity. The coordinate ξ^4 represents the time t multiplied by the constant speed of light in vacuum c such that ξ^4 has dimension of length. Other sets of coordinates could be indifferently chosen to parameterize the points of the manifold. Consider two possible sets of coordinates noted ξ^{μ} and $\widetilde{\xi}^{\mu}$. The transformation of the coordinates is given by:

$$d\widetilde{\xi}^{\mu} = \frac{\partial \widetilde{\xi}^{\mu}}{\partial \xi^{\nu}} d\xi^{\nu} \tag{2}$$

where $\frac{\partial \widetilde{\xi}^{\mu}}{\partial \xi^{\nu}}$ is the Jacobian matrix of the coordinate transformation; the determinant of this Jacobian matrix is denoted $\left|\frac{\partial \widetilde{\xi}^{\mu}}{\partial \xi^{\nu}}\right|$. Einstein's summation convention is used throughout this article and upper indices denote contravariant quantities while lower indices denote covariant quantities.

An interval ds is defined as a generalized length element, such that:

$$ds^2 = -(d\zeta^1)^2 - (d\zeta^2)^2 - (d\zeta^3)^2 + (d\zeta^4)^2 \tag{3}$$

where the coordinates ζ^{μ} represent the 4D coordinates of an event in a 4D cartesian coordinate system. This particular coordinate system is also said to be standard, Galilean or inertial and the components of the metric tensor are in this case:

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}. \tag{4}$$

It is supposed that such a set of coordinates exists for every point of the manifold. As a 4-scalar, the interval ds is constant under any change of 4D coordinate systems. Using Eqs. 2 and 3, the covariant components $g_{\mu\nu}$ of the metric tensor for the coordinate system ξ^{μ} are defined as:

$$g_{\mu\nu} = \frac{\partial \zeta^{\lambda}}{\partial \xi^{\mu}} \frac{\partial \zeta^{\kappa}}{\partial \xi^{\nu}} \eta_{\lambda\kappa} \tag{5}$$

The covariant and contravariant components of the metric tensor are such that:

$$g_{\mu\lambda} g^{\lambda\nu} = g^{\nu}_{\mu} = \delta^{\nu}_{\mu} \tag{6}$$

where δ^{ν}_{μ} is the Krönecker's symbol.

A set of four base vectors is further defined associated to the 4D coordinate system ξ^{κ}. Any set of four vectors $g_{\mu}(\xi^{\kappa})$ (covariant) or $g^{\mu}(\xi^{\kappa})$ (contravariant) representing the local base vectors, can be changed from the set of orthonormal base