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# Singularities and wrinkling: The case of a concentrated force

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ABSTRACT

The edge wrinkling of a uniformly stretched circular elastic plate subjected to a central concentrated load is considered within the framework of the Föppl–von Kármán nonlinear plate theory. Singular perturbation methods are employed to obtain a three-term asymptotic formula for the critical load in terms of a non-dimensional quantity that depends on the initial pre-stress. Comparisons between the analytical predictions and direct numerical simulations of the full bifurcation eigenproblem provide further insight into the accuracy and limitations of the derived results.

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## 1. Introduction

Unlike rigid-body mechanics, where concentrated forces are ubiquitous, in linear elasticity the use of such loads tends to be more restricted because of the singular nature of the local stresses associated with this particular type of situation (see, for example, Sinclair, 2004). Apart from their undisputed usefulness as mathematical techniques for obtaining the Green's function of various unbounded configurations (e.g. Kachanov, Shafiro, & Tsukrov, 2003), concentrated forces remain somewhat problematic because the assumptions of linear elasticity are invariably violated near the points where they are applied. The usual device employed to circumvent this difficulty is an appeal to Saint Venant's principle (Sternberg & Eubanks, 1955; Turteltaub & Sternberg, 1968).

In structural mechanics there are many situations involving thin-walled bodies for which point-loads are unavoidable. For instance, if the area over which a surface load is distributed is smaller than the wall thickness, it is advantageous to model the load as a concentrated force or moment acting at a single point. There has been much effort devoted to these types of problems and comprehensive reviews have been given by Lukasiewicz (1976, 1979), which contain useful additional information. An example of a practical application in which concentrated loads play an important role is the shaft-loaded blister test used for measuring the interfacial fracture energy between a coating film and a substrate (Jin & Wang, 2008; Wan & Mai, 1995; Zhao, Zheng, & Fan, 2010).

The accurate description of the buckling of thin flat plates loaded by in-plane concentrated forces poses a significantly more challenging problem than the usual configuration in which uniformly distributed tractions act along the perimeter of the plate (e.g., see Leissa & Ayoub, 1989; Ventsel & Krauthammer, 2001). Most of the complications can be traced to the task of determining the inhomogeneous stress distribution associated with in-plane localised forces. Standard series solutions are frequently impractical because they are often slowly convergent and require an inordinately large number of

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terms to obtain sufficiently accurate results (Hopkins, 1949; Leggett, 1937). Alfutov and Balabukh (1967, 1968) designed an ingenious approximate technique without the need to actually solve for the exact pre-buckling stresses. A recent review of the method has been given by Platt, Davies, and Snell (1992) and it is unfortunate that this expedient strategy is not applicable for out-of-plane loading as well.

It has been known for some time that *weakly supported* plates tend to experience a somewhat counter-intuitive form of buckling when loaded transversely. Typically, the instability pattern is confined to a relatively narrow zone near the edge of the plate. The focus of our study here is with such edge-buckling of a circular thin elastic plate that is first stretched uniformly before a normal point-load is applied at its centre. Since buckling occurs here in the presence of "tensile loading" we shall adopt the standard current terminology and describe this bifurcation phenomenon as *edge-wrinkling*.

Within the framework of Föppl–von Kármán (FvK) kinematics, the deformation of circular elastic plates under lateral loads has a long history in solid mechanics. Of particular interest has been a description of how the central transverse displacement varies with applied load. Early attacks on this question used power-series solutions summarised in the book by Chia (1980), although more recently Cao (1996, 1997) has provided improved versions incorporating techniques to accelerate the rate of convergence (e.g., Van Dyke, 1974). In general, for uniform pressure loads no simple closed-form solutions exist, but in his classic work Schwerin (Schwerin, 1929) noted an exception for the case of an (unstretched) circular elastic membrane subjected to a concentrated normal force at its centre, when the Poisson's ratio is equal to $\frac{1}{3}$ (see also Mansfield, 2008). Jahsman, Field, and Holmes (1962) extended Schwerin's solution to include background tension and obtained implicit-form results valid for arbitrary Poisson's ratios. Other implicit solutions have been obtained by Nachbar (1968) for a wider range of geometries and types of concentrated loads (e.g., annular membranes, ring loads, etc.). Frakes and Simmonds (1985) extended Schwerin's solution using asymptotic methods, while Komaragiri, Begley, and Simmonds (2005) established the form of Reissner's simplified plate equations in the various asymptotic limits, for both pressure and point loads.

The edge wrinkling of a stretched circular plate subjected to a concentrated central load was originally investigated by Adams (1993) using the FvK plate theory in conjunction with a dynamical buckling approach (e.g., see Bolotin, 1963). His solution was entirely numerical and involved examining the asymmetric frequency spectrum corresponding to small free vibrations of the plate in the neighbourhood of the nonlinear axisymmetric equilibrium state. The buckling mode and corresponding critical load were deduced by observing the vanishing of the lowest frequency. A very similar and much earlier work is that of Archer and Famili (1964), who were concerned with the edge wrinkling of an externally pressurised shallow spherical cap. Although Adams (1993) correctly found that the eigenmodes of his equations are localised near the edge of the plate, his solution also contains a somewhat mysterious fast oscillatory component in the transverse displacement, whose origin has remained unclear. Some of our preliminary numerical work based on the same equations, but using the standard static approach to identify the wrinkling loads (Brush & Almroth, 1975; Ventsel & Krauthammer, 2001) did not exhibit such an oscillation. Part of the motivation of the present study is to probe possible explanations for this discrepancy by examining more closely the asymptotic structure of the bifurcation equations for a stretched circular plate subjected to a point-load at its centre. Since this problem is conservative, one would expect the dynamical and static approaches to be equivalent, and in what follows we pursue the latter path.

As will become readily apparent as we proceed, the nonlinear basic state plays a central role in fixing the structure of the edge wrinkling. Bearing this in mind, we start the paper in Section 2 with a brief overview of the axisymmetric solution for the Föppl–von Kármán system. In particular, we discover that the wrinkling problem itself depends only on two dimensionless parameters $\lambda$ and $\mu$ which relate to the magnitude of the concentrated load and the background tension, respectively. In Section 3, the method of adjacent equilibrium is used to linearise the full nonlinear FvK equations about the axisymmetric solution developed in Section 2. The properties of the resulting eigenproblem are then illustrated by a representative sample of direct numerical simulations, which confirm the localised nature of the wrinkling pattern and provide several clues about the form of the basic state. In Section 4 we address the asymptotic structure of the basic state when $\mu \gg 1$, while the corresponding analysis for the bifurcation equation appears in Section 5. We include comparisons between our main asymptotic predictions and direct numerical solutions of the full wrinkling problem, which confirm the accuracy of the results obtained. An interesting question that arises in our problem is the size of the transverse displacement of the centre of the plate; this issue is investigated in Section 6. Finally, the paper closes with a discussion of our main findings and looks forward to possible extensions.

## 2. Formulation: the basic state

Consider a circular thin elastic plate of uniform thickness $h > 0$ and radius $a \gg h$, which is initially stretched uniformly before a concentrated load is applied at its centre. Along its circumference the plate is flexurally clamped, while the in-plane boundary conditions correspond to prescribed radial tractions as shown in Fig. 1. The deformation of this configuration is expressed with respect to an usual cylindrical system of coordinates $(r, \theta, z)$ with the $z$-axis perpendicular to the median plane of the plate, which also contains the origin of the axes. The linearly elastic material of the plate is characterised by the Youngs' modulus $E$ and the Poisson's ratio $\nu$.

The starting point for the development of the relevant bifurcation problem is the well-known Föppl–von Kármán (FvK) equations (e.g., see Brush & Almroth, 1975; Ventsel & Krauthammer, 2001) written in terms of the transverse displacement

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