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A unified integro-differential nonlocal model

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ABSTRACT

In this paper a unified integro-differential nonlocal elasticity model is presented and its use in the bending analysis of Euler–Bernoulli beams is illustrated. A general (for an elastic continuum) finite element formulation for the two-phase integro-differential form of Eringen nonlocal model is provided. The equations are specialized for the case of the Euler–Bernoulli beam theory. Several numerical examples, including the paradoxical cantilever beam problem that eluded other researchers, are provided to show how the present nonlocal model affects the transverse displacement of beams. The examples show that Eringen nonlocal constitutive relation has a softening effect on the beam, except for the case of the simply supported beam. A brief discussion on the applicability of the integro-differential model to other problems is also presented. Finally, the transition from the stiffened nonlocal simply supported beam to the softened nonlocal clamped beam is also investigated.

1. Introduction

For hyperelastic materials (i.e., Green elastic materials) there exists a potential function whose derivative with respect to the strain at a point gives the corresponding stress at that point (Reddy, 2013). This forms the basis for local (conventional) constitutive model where the stress and strain at each point are related. Local theory of continuum mechanics is inherently scale free, i.e. forces are only transferred through contact and no long-range forces between points located further apart is considered. However, there exists certain phenomena (e.g., dispersion of elastic waves, crack propagation in fracture mechanics, dislocations, and so on) that cannot be explained using local theory of elasticity. In addition, as a consequence of recent developments in the field of material science there is a need to model the structural response of a variety of new materials that require the consideration of nonlocal aspects of the material (e.g. size effect in nanomaterials). In nonlocal theories, stress at each point is influenced by the strain at all points in the domain. This influence decreases as the distance between the points increases. The concept of nonlocal theory of linear elasticity was initially introduced in papers by Kröner (1967), Krumhansl (1968) and Kunin (1968). Later, the idea of long-range interactions was further developed in the works of Eringen (1972b, 1972a, 1983, 2002) and Eringen and Edelen (1972). Eringen (1983) introduced an integro-differential nonlocal model which has widely been used in the literature. Later, Eringen proposed a two-phase nonlocal model (Eringen, 1987) which was a combination of local and integro-differential nonlocal constitutive theories. One of the advantages of an integral nonlocal theory over the local elasticity theory is that the former gives non-singular results for geometric singularities (i.e. cracks) due to the averaging effect inherent in the integral form of the constitutive relation.

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The nonlocal integral constitutive equation makes use of a positive distance-decaying kernel function which specifies the dependence of stress at each point on the strain at other points in the domain. Eringen (1983) showed that for a specific class of kernel functions the Eringen nonlocal integral constitutive equation can be transformed into a differential form with the exact same properties. Due to the difficulties in using integral constitutive equations, the nonlocal differential model proposed by Eringen (1983) is the one most widely used in the literature to account for nonlocal effects. Several studies have been reported on the basis of nonlocal theories. Peddieson, Buchanan, and McNitt (2003) used the Eringen nonlocal differential model to derive the equations of equilibrium for a nonlocal Euler–Bernoulli beam. This study (Peddieson et al., 2003) was pioneering in the sense that Eringen nonlocal differential model was used to incorporate nonlocal effects into the analysis of structural elements. One of the main issues that was discussed in the work of Peddieson et al. (2003) was the fact that in nonlocal cantilever beams (enhanced with Eringen's differential model) nonlocal effects were not triggered for point loads applied at the free end. This is not a desirable outcome, because recently cantilever beams of micro- and nano- size have found several applications as actuators and sensors in the field of chemical and biological sciences (Ekinci & Roukes, 2005; Lavrik, Sepaniak, & Datskos, 2004; Pei, Tian, & Thundat, 2004; Pereira, 2001). If a nonlocal model is not capable of capturing the size effect in these nano- and macro-cantilever beams, then the data obtained by these devices may not be interpreted correctly.

Other examples of nonlocal Euler–Bernoulli beam studies were presented in Sudak (2003), Challamel and Wang (2008), Lu, Lee, Lu, and Zhang (2006) and Shakouri, Lin, and Ng (2009). Challamel and Wang (2008) also pointed to the deficiency mentioned in Peddieson et al. (2003) and suggested the integration of gradient elasticity model with Eringen nonlocal model to eliminate it. Shakouri et al. (2009) gave a discrete formulation for nonlocal Euler–Bernoulli beam representation of the double-walled carbon nanotubes using the Galerkin method. Wang, Kitipornchai, Lim, and Eisenberger (2008, 2006), Wang and Wang (2007) and Wang and Liew (2007) integrated Timoshenko beam theory with Eringen nonlocal model. The main problem with these works (Wang et al., 2008; Wang & Liew, 2007; Wang & Wang, 2007; Wang et al., 2006) is that nonlocal effects are only limited to normal stresses and not transverse shear stresses. Reddy (2007) used Eringen nonlocal model to give the variational statements for several beam theories, namely the Euler–Bernoulli, Timoshenko, Reddy and Levinson beam theories. In this comprehensive study (Reddy, 2007) the above-mentioned limitation imposed in the works of Wang et al. (2008), Wang et al. (2006), Wang and Liew (2007) and Wang and Wang (2007) is removed and nonlocal effects are included in both normal and transverse shear stresses. Analytical solutions of static bending, vibration, and buckling of the beams are also provided in this study. Later, Reddy (2010) formulated the governing equations for the bending of beams (Euler–Bernoulli and Timoshenko beam theories) and plates (Classical and first order shear deformation plate theories) which also took in account von Kármán nonlinearity. Reddy (2010) stated that no quadratic functional can be derived for the differential form of Eringen nonlocal beam theory from which the governing equations can be derived. Thai (2012) and Thai and Vo (2012) recently provided a higher order nonlocal beam theory which is slightly different from Reddy beam theory which also accounted for variation of shear stress along the height of the beam. Reddy and El-Borgi (2014) provided the governing equations for bending of nonlocal Euler–Bernoulli and Timoshenko beam theories accounting for moderate rotations through modified von Kármán nonlinearity. Several studies have also applied Eringen nonlocal model to the study of functionally graded beams (Reddy, El-Borgi, & Romanoff, 2014; Rahmani & Pedram, 2014; Salehipour, Shahidi, & Nahvi, 2015). Studies on nonlocal beam theories based on the differential model are far more exhaustive to be reported here. Interested readers may consult (Reddy & El-Borgi, 2014; Reddy, El-Borgi, & Romanoff, 2014).

In all of the above-mentioned references, the differential form of the Eringen model had been used. Polizzotto (2001) applied the integral form of Eringen model and derived the variational principles governing the integral form from which the nonlocal finite element formulation is obtained. The kernel function in the integral constitutive equation brings in a concept of a length scale. Pisano and Fuschi (2003) used the approach proposed by Polizzotto (2001) to derive a closed-form solution for a bar in tension with nonlocal Eringen model as the constitutive equation. Later, Pisano, Sofi, and Fuschi (2009) used this integro-differential nonlocal model to give a finite element formulation for 2D problems of two-phase elastic materials (Eringen, 1987). Di Paola, Failla, Sofi, and Zingales (2011) came up with a new method to introduce long-range forces into the equations of motion. General 3D variational statements were constructed and they were further simplified for the Timoshenko beam theory. The formulation proposed by Di Paola et al. (2011) is conceptually similar to the formulation of peridynamic theory proposed by Silling (2000).

It is found by several authors that Eringen's differential model yields inconsistent results for a cantilever when compared to other boundary conditions (Challamel & Wang, 2008; Challamel et al., 2014; Peddieson et al., 2003; Wang & Liew, 2007; Wang et al., 2008). For all boundary conditions except the cantilever, the model predicts softening effect (i.e., larger defections and lower fundamental frequencies) as the nonlocal parameter is increased. Several ad hoc approaches or explanations have been proposed to alleviate the baffling case of the cantilever beam. In the present study, classical theory of elasticity is augmented with Eringen's nonlocal model in integral form to present a unified integro-differential model for nonlocal elasticity and a general finite element formulation for the integral form of Eringen nonlocal model. Note that by using the two-phase Eringen model (Eringen, 1987), two control parameters will exist, namely the length scale parameter and phase parameter. The general 3D equations are further simplified to the one-dimensional case of the Euler–Bernoulli beam theory. Several examples are provided to show how Eringen nonlocal model affects the transverse displacement of the beams. In this study, the kernel function used in the integral constitutive equation is different from that of which yields into Eringen's differential equation (Eringen, 1983). It is shown that the proposed nonlocal model yields consistent results for most boundary conditions (including the paradoxical case of a cantilever beam), however, the results are slightly different for the case of a