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Bogus transformations in mechanics of continua

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ABSTRACT

In this paper we consider the structure of the symmetry group of some important mechanical theories (nonlinear elasticity and fluids of grade *n*). We discuss why the invariance with respect to some well-known transformations must be used with care and we explain why some of these universal transformations are useless to obtain invariant solutions of physical significance.

1. Introduction

Lesson four of Giancarlo Rota's invited address delivered at the meeting of the Mathematical Association of America in 1997, today known by the title *Ten lessons I wish I had learned before I started teaching differential equations*,[1] is to *Teach changes of variables*. Rota writes:

> *Worse, no one realizes that changes of variables are not just a trick; they are a coherent theory (it is the differential analogue of classical invariant theory, but let it pass).*

We think that Rota has been taken literally by a large community of applied mathematicians. Today, it is possible to record several books entirely devoted to symmetry groups applied to differential equations (Bluman & Kumei, 1989; Bocharov et al., 1999; Cantwell, 2002; Ibragimov, 1993; Olver, 1993; Stephani, 1990), a long list of review papers and a tremendous quantity of scientific papers. This means that we have at our disposal a full catalogue of symmetries for a large class of differential equations.

It is well known that the computation of the classical transformations admitted by differential equations is a completely algorithmic procedure and this procedure may be automatized using symbolic software. The reason is that Lie's method is extremely powerful but, on the other hand, this is indeed also the origin of a major drawback: automatism produces an *abuse* of the methodology with respect to the understanding of the problem under investigation.

Today, many of the published papers about symmetries of differential equations are quite unsatisfactory. Some papers start from an equation without any knowledge of its physical motivations, then they provide a list of symmetries (whose computation sometimes may be considered as just a big exercise) and contain some solutions. These solutions, often, are non sense from the

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[1] See http://www.math.toronto.edu/lgoldmak/Rota.pdf .

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physical standpoint (just mathematical curiosity) and in many cases they are displayed without any discussion on their possible meaning in the context of the mechanical theory investigated.

For people seriously interested in Mechanics the simple knowledge of the symmetries and transformations admitted by a given model is of no interest without an investigation of the physical meaning of this invariance. On the other hand, the standard situation that we observe concerning point symmetries of a given mechanical or physical theory may be very disappointing: the full group of transformations admitted by the differential equations describing the given theory may be guessed by a simple inspection of the basic principles underlying the theory itself. This is the case of uniformity of the material properties, frame indifference and material symmetry, as seen in many textbooks (e.g. Antman, 1995; Dafermos, 2005; Truesdell and Rajagopal, 2010). Physical intuition and experience are enough to discover the fundamental transformations but clearly only the general theory may be give us the complete picture.

In Edelen (1982) we read

> *The isovector[2] fields of the incompressible Navier–Stokes equations thus generate the already known transformations admitted by those equations. What is new is the fact that the isovector method is exhaustive; there are no other mappings admitted by the incompressible Navier–Stokes equations.*

The above remark by Edelen can be recast in a more essential but communicative language: *nothing new is under the sun but now you know it for sure*.

The knowledge of the symmetry group of a mechanical theory is always of interest if this knowledge is coupled with a clear understanding of these symmetries to uncover the nature of the transformations and the usefulness of their mathematical properties.

Some of the symmetries underlying a mechanical theory are fundamental bricks in the construction and characterization of the theory itself, but if such a transformation is used to build exact solutions using reduction methods they are a sort of *bogus* transformation. They are useless because they generate solutions of no mechanical interest. The aim of the present note is to develop this point.

Our arguments are based on general considerations and two basic theories of continuum mechanics: the theory of nonlinear elasticity and the Navier–Stokes equations. General considerations are based on frame-indifference as a basic principle for continuum mechanics that provides the fundamental symmetries of the theory. Frame-indifference has several analogies with the symmetries that are postulated in gauge theories. Then, we specialize our arguments to the above two theories. The fact that the theories are well known will help the reader in understanding our points.

2. Basic equations

For the sake of simplicity here we are interested in purely mechanical theories of non polar materials. Therefore, let $\boldsymbol{x}$ denote the current position of a particle $\boldsymbol{X}$ in the reference configuration that is assumed to be stress free. The motion of the body is a one-to-one mapping $\boldsymbol{\chi}(\boldsymbol{X}, t)$ that assigns to each point $\boldsymbol{X}$ belonging to the reference configuration the position $\boldsymbol{x}$ at time t, i.e. $\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}, t)$. We make the hypothesis of the existence of a functional $\mathcal{F}$ such that we have the following expression for the stress $\boldsymbol{T}$ at time t:

$$\boldsymbol{T} = \mathcal{F}(\boldsymbol{\chi}). \tag{2.1}$$

A basic principle of continuum mechanics, of interest in what follows, is that physical laws be independent of the frame of reference. Usually this principle is denoted as *frame-indifference* (see e.g. Truesdell & Rajagopal, 2010). Given a process $\{\boldsymbol{\chi}, \boldsymbol{T}\}$ and a process $\{\boldsymbol{\chi}^*, \boldsymbol{T}^*\}$ related by

$$\boldsymbol{\chi}^*(\boldsymbol{X}, t^*) = \boldsymbol{Q}(t)\boldsymbol{\chi}(\boldsymbol{X}, t) + \boldsymbol{c}(t), \quad \boldsymbol{T}^*(\boldsymbol{X}, t^*) = \boldsymbol{Q}(t)\boldsymbol{T}(\boldsymbol{X}, t)\boldsymbol{Q}(t)^T,$$

and $t^* = t - a$. Here $\boldsymbol{Q}(t)$ is a rotation and $\boldsymbol{c}(t)$ is an arbitrary point and a arbitrary number. We require the *indifference* (i.e. invariance) of the constitutive equation for $\mathcal{F}$ with respect rigid translations, shifting of the time scale and rigid rotations i.e.

$$\boldsymbol{Q}\mathcal{F}(\boldsymbol{\chi})\boldsymbol{Q}^T = \mathcal{F}(\boldsymbol{Q}\boldsymbol{\chi}). \tag{2.2}$$

We stress that frame indifference is a broader invariance than the usual Galilei group invariance, as the orthogonal transformation $\boldsymbol{Q}$ depends on time and the translation vector $\boldsymbol{c}$ may not depend linearly on time in general. The invariance group of the frame-indifference is thus the infinite-dimensional group of functions

$$(\boldsymbol{Q}, \boldsymbol{c}) : \mathbb{R} \to SO(3) \times \mathbb{R}^3. \tag{2.3}$$

The class of field theories which are characterized by invariance groups of a similar structure is the class of *gauge theories*. Mathematically, any theory which possess an infinite-dimensional group of symmetries can be regarded as a gauge theory (Bocharov et al., 1999). This definition encompasses not only continuum mechanics but also general relativity and the theory of electro-weak–strong interactions which underlies the Standard Model of particle physics.

The equations of general relativity were derived by Einstein by prescribing a gauge group of symmetries (or, equivalently, by requiring the general covariance of the theory) and by requiring the existence of distinguished conservation laws. In Yang-Mills

[2] The isovector method is a method to compute symmetries of differential equations based on the use of external differential forms.