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Bending of Euler–Bernoulli beams using Eringen's integral formulation: A paradox resolved

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ABSTRACT

The Eringen nonlocal theory of elasticity formulated in differential form has been widely used to address problems in which size effect cannot be disregarded in micro- and nano-structured solids and nano-structures. However, this formulation shows some inconsistencies that are not completely understood. In this paper we formulate the problem of the static bending of Euler–Bernoulli beams using the Eringen integral constitutive equation. It is shown that, in general, the Eringen model in differential form is not equivalent to the Eringen model in integral form, and a general method to solve the problem rigorously in integral form is proposed. Beams with different boundary and load conditions are analyzed and the results are compared with those derived from the differential approach showing that they are different in general. With this integral formulation, the paradox that appears when solving the cantilever beam with the differential form of the Eringen model (increase in stiffness with the nonlocal parameter) is solved, which is one of the main contributions of the present work.

1. Introduction

The formalism based on the classical continuum mechanics has been widely used to develop powerful and reliable simulation tools to solve fundamental problems in several engineering fields such as civil, mechanical, aerospace, biomedical as well as in other applications of physical sciences. A basic feature of the local theory of continuum mechanics is that the stress at each point is related to the strain at the same point only. Therefore, a defining characteristic of this framework is that it is scale-free.

However, the matter is discrete and heterogeneous in nature. Materials used nowadays, like composites, functionally graded materials, polycrystalline solids, granular materials, and so on, all have inherent microstructures at different scales. Additionally, at high-frequency excitations, microstructural and size effects are observed in wave propagation in solids when the wavelength of a traveling wave becomes comparable with the scale of material heterogeneities (Gonella, Greene, & Liu, 2011). Moreover, modern technological applications involve the use of systems which can be devised as micro- or nano-structures, mainly in micro- or nano-electromechanical (MEMS or NEMS) devices (Martin, 1996), nano-machines (Bourlon, Glattli, Miko, Forro, & Bachtold, 2004; Drexler, 1992; Fennimore et al., 2003; Han, Globus, Jaffe, & Deardorff, 1997), as well as in biotechnology and biomedical fields (Saji, Choe, & Young, 2010). A main characteristic of these nanostructures is that their dimensions become comparable to the microstructural characteristics distances, thus the size effects are significant regarding their mechanical behavior.

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The above problems could be addressed using molecular dynamics (He, Liew, & Wei, 2007; Liew, Wei, & He, 2007; Tsai & Fang, 2007; Wei & Srivastava, 2004), but this approach requires a great computational effort, providing a motivation towards developing higher-order and nonlocal continuum mechanics theories able to capture the size effects by introducing intrinsic lengths in their formulations. Therefore, classical continuum mechanics, due to its inherent scale-free characteristic, cannot predict the size effect present in the above mentioned applications.

Apart from the first attempts to capture scale effects using the continuum theories through the works of Cauchy and Voigt in the 19th century, and the work of the Cosserat brothers in the first half of the 20th century, contributions by Mindlin and Tiersten (1962), Kröner (1963, 1967), Toupin (1963, 1964), Green and Rivlin (1964), Mindlin (1964, 1965), Krumhansl (1968), Mindlin and Eshel (1968), Kunin (1968), Eringen (1972a, 1972b), Eringen and Edelen (1972), constitute a major revival of the nonlocal and higher-order theories. The concept of nonlocal theory of linear elasticity was initially introduced by Kröner (1967), Krumhansl (1968), and Kunin (1968), and further developed by Eringen (1972a, 1972b), and Eringen and Edelen (1972). The basic feature of the nonlocal theories of elasticity is that the stress at each point is related to the strain at all points in the domain. This influence decreases as the distance between the point of interest and the neighboring points increases. The Eringen nonlocal integral constitutive equation describes the dependence of the stress at a point on the strain in the rest of the domain through a positive-decaying kernel function.

A differential constitutive theory, introduced by Eringen (1983), showed that for a specific class of kernel functions the nonlocal integral constitutive equation can be transformed into a differential form, much easier to manage than the integral model. From the pioneering work of Peddieson, Buchanan, and McNitt (Peddieson, Buchanan, & McNitt, 2003), and due to its simplicity, this differential Eringen nonlocal model has been widely used to analyze the static, buckling, and dynamic behavior of nanostructures. The list of papers is extremely long to be reported here. Nevertheless we cite some representative works.

As stated before, the Eringen nonlocal theory of elasticity formulated in differential form has been used to address the behavior of linear beams (Ke, Wang, & Wang, 2012; Loya, Lopez-Puente, Zaera, & Fernandez-Saez, 2009; Lu, 2007; Reddy, 2007; Wang, Zhang, & He, 2007; Wang, Zhang, Ramesh, & Kitipornchai, 2006; Xu, 2006), beams with von Kármán nonlinearity (Reddy, 2010; Reddy & El-Borgi, 2014), functionally graded beams (H. Salehipour, 2015; O. Rahmani, 2014; Reddy, El-Borgi, & Romanoff, 2014), beams under rotation (Aranda-Ruiz, Loya, & Fernández-Sáez, 2012; Murmu & Adhikari, 2010b; Narendar & Gopalakrishnan, 2011b; Pradhan & Murmu, 2010), rods (Kiani, 2010; Murmu & Adhikari, 2010a; Murmu & Pradhan, 2009b; Narendar, 2011; Narendar & Gopalakrishnan, 2010; Sun & Zhang, 2003), plates (Hosseini-Hashemi, Zare, & Nazemnezhad, 2013; Ke, Wang, & Wang, 2008; Murmu & Pradhan, 2009a), plates with von Kármán nonlinearity (Reddy, 2010), cylindrical shells (Hua, Liew, Wang, He, & Yakobson, 2008; Wang & Varadan, 2007; Wang & Wang, 2007), conical shells (Firouz-Abadi, Fotouhi, & Haddadpour, 2011; Liew et al., 2007; Tsai & Fang, 2007), rings (Moosavi, Mohammadi, Farajpour, & Shahidi, 2011; Wang & Duan, 2008), spherical shells (Ghavanloo & Fazelzadeh, 2013a; Vila, Zaera, & Fernandez-Saez, 2015; Zaera, Fernandez-Saez, & Loya, 2013), and particles (Ghavanloo & Fazelzadeh, 2013b), as well as carbon nanotubes (CNTS) (Ansari, Shahabodini, & Rouhi, 2013; Chen, Lee, & Eskandarian, 2004; Fleck & Hutchinson, 1997; Heireche, Tounsi, Benzair, Maachou, & Adda Berdia, 2008; Murmu & Pradhan, 2009c; Narendar & Gopalakrishnan, 2011a; Sudak, 2003; Zhou & Li, 2001).

Nevertheless, several authors have pointed out the inconsistent results obtained from the Eringen differential model regarding a cantilever beam when compared to other boundary conditions (Challamel & Wang, 2008; Challamel et al., 2014; Peddieson et al., 2003; Wang, Kitipornchai, Lim, & Eisenberger, 2008; Wang & Liew, 2007). For all boundary conditions except the cantilever, the model predicts softening effect (i.e. larger deflections and lower fundamental frequencies) as the nonlocal parameter is increased. Moreover, Lu, Lee, Lu, and Zhang (2006) showed that, depending on the nonlocal parameter, it is only possible to calculate a few natural frequencies of flexural vibrations of a cantilever beam. This last finding may be a consequence of the non self-adjoint characteristic of the Eringen differential operator (Challamel et al., 2014; Reddy, 2007).

Benvenuti and Simone (2013) found also inconsistent results regarding the behavior of a bar in tension. They observed that nonlocal solutions based on differential form of the Eringen theory are not consistent with the constitutive equation formulated in integral form. The reason is that in the transformation process of the constitutive equations from integral to differential forms, certain boundary conditions are not properly fulfilled (Polyanin & Manzhirov, 2008)

To overcome this paradoxical behavior, Challamel and Wang (2008) proposed a local/nonlocal moment-curvature relationship. In the same way, Challamel, Rakotomanana, and Le Marrec (2009) adopted a mixed local/nonlocal model to address the 1D wave propagation. In order to fit the dispersion relations obtained from a Von-Kármán lattice, Challamel et al. (2009) selected a negative value for the "mixture" parameter leading to thermodynamic inconsistencies (Fafalis, Filopoulos, & Tsamasphyros, 2012). On the other hand, the two-phase constitutive model with both local and nonlocal phases was early proposed by Eringen (1987). This theory was further developed by Polizzotto (2001) who derived the variational principles governing the integral form. Using this two-phase constitutive model, Pisano and Fuschi (2003) and Benvenuti and Simone (2013) solved the problem of a bar in tension, and very recently Khodabakhshia and Reddy (2015) have presented the analysis of the static bending of Euler–Bernoulli beams subjected to different boundary and load conditions.

In this paper we formulate the problem of the static bending of Euler–Bernoulli beams using the Eringen integral constitutive equation. It is showed that, in general, the Eringen model in differential form is not equivalent to the Eringen model in integral form. Although this has been shown for a particular case (bending of beams), it also applies to other problems. A general method to solve rigorously the problem in integral form is proposed. Different boundary and load conditions are analyzed and the results have been compared whose derived from the widely used differential approach showing that they are different in general. With this integral formulation, the paradox that appears when solving the cantilever beam with the differential form of the Eringen model (increase in stiffness with the nonlocal parameter) is solved, being this one of the main outcomes of the work.