International Journal of Engineering Science 97 (2015) 18–25

Contents lists available at ScienceDirect

International Journal of Engineering Science

journal homepage: www.elsevier.com/locate/ijengsci

On the independence of strain invariants of two preferred direction nonlinear elasticity

M.H.B.M. Shariff[a], R. Bustamante[b,*]

[a] Khalifa University of Science, Technology and Research, United Arab Emirates
[b] Departamento de Ingeniería Mecánica, Universidad de Chile, Beauchef 851, Santiago Centro, Santiago, Chile

ARTICLE INFO

Article history:
Received 11 November 2014
Revised 8 June 2015
Accepted 4 August 2015
Available online 4 September 2015

Keywords:
Principal axis
Syzygies
Two-fibres solids

ABSTRACT

It is often assumed in the literature that the nine classical strain invariants, which are used to characterize the strain energy of a compressible anisotropic elastic solid with two preferred non-orthogonal directions are independent. In this paper, it is shown that only six of the classical strain invariants are independent, and syzygies exist between the classical invariants. Alternatively, using principal axis techniques, it is simply proven that, only six of the classical strain invariants are independent and syzygies exist between the principal axis strain invariants. Consequently, all other sets of strain invariants, proposed in the literature, which are uniquely related to the set of principal axis strain invariants, have only six independent invariants. Due to syzygies, it is shown that the number of ground state constants required to fully describe the quadratic linear strain energy function of two-fibre solids is fourteen, not thirteen, as assumed in the literature.

1. Introduction

Following the work of Spencer (1984), a strain energy function W_F of a compressible elastic material with two preferred unit directions $\boldsymbol{a}$ and $\boldsymbol{b}$ can be expressed as

$$W_F = W(\boldsymbol{C}, \boldsymbol{a}\otimes\boldsymbol{a}, \boldsymbol{b}\otimes\boldsymbol{b}), \tag{1}$$

where $\boldsymbol{C}$ is the right Cauchy–Green deformation tensor and $\otimes$ denotes the dyadic product. W is an isotropic invariant function of $\boldsymbol{C}, \boldsymbol{a}\otimes\boldsymbol{a}$ and $\boldsymbol{b}\otimes\boldsymbol{b}$, i.e.,

$$W(\boldsymbol{C}, \boldsymbol{a}\otimes\boldsymbol{a}, \boldsymbol{b}\otimes\boldsymbol{b}) = W\left(\boldsymbol{QCQ}^T, \boldsymbol{Q}(\boldsymbol{a}\otimes\boldsymbol{a})\boldsymbol{Q}^T, \boldsymbol{Q}(\boldsymbol{b}\otimes\boldsymbol{b})\boldsymbol{Q}^T\right) \tag{2}$$

must be satisfied for all proper orthogonal tensors $\boldsymbol{Q}$. It follows that the strain energy function W_e can be expressed in terms of a set of invariants

$$\mathcal{S}_B = \{I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8, I_9, I_{10}\}, \tag{3}$$

where

$$I_1 = \mathrm{tr}(\boldsymbol{C}), \quad I_2 = \frac{I_1^2 - \mathrm{tr}\left(\boldsymbol{C}^2\right)}{2}, \quad I_3 = det(\boldsymbol{C}), \quad I_4 = \boldsymbol{a}\bullet\boldsymbol{Ca}, \tag{4}$$

* Corresponding author. Tel.: +56 2 29784543; fax:+56-2-6896057.
E-mail address: rogbusta@ing.uchile.cl (R. Bustamante).

http://dx.doi.org/10.1016/j.ijengsci.2015.08.002
0020-7225/

$$I_5 = \boldsymbol{a} \bullet \boldsymbol{C}^2\boldsymbol{a}, \quad I_6 = \boldsymbol{b} \bullet \boldsymbol{C}\boldsymbol{b}, \quad I_7 = \boldsymbol{b} \bullet \boldsymbol{C}^2\boldsymbol{b}, \quad I_8 = (\boldsymbol{a} \bullet \boldsymbol{b})\boldsymbol{a} \bullet \boldsymbol{C}\boldsymbol{b}, \tag{5}$$

$$I_9 = (\boldsymbol{a} \bullet \boldsymbol{b})^2 \neq 0, \quad I_{10} = (\boldsymbol{a} \bullet \boldsymbol{b})\boldsymbol{a} \bullet \boldsymbol{C}^2\boldsymbol{b} \tag{6}$$

and tr denotes the trace of a second order tensor. The invariant I_9 is independent of strain and hence the set

$$\mathcal{S}_C = \{I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8, I_{10}\} \tag{7}$$

of 9 invariants is commonly used to describe the strain energy function (see Spencer, 1984). In this paper, we show that only seven of the the 10 invariants in (3) are independent or six of the nine invariants in (7) are independent. In the case when the preferred directions are orthogonal, $I_8 = I_9 = I_{10} = 0$, Shariff (2013) has shown that only six of the seven invariants I_j, $j = 1, 2, 3, \ldots, 7$ are independent. In Section 2, the proof is presented using a set of principal axis invariants, while in Section 3 the proof is done directly using the definition of the invariants given in (4)–(6). In Section 4, the consequences of the syzygies on the number of ground state constants are discussed via linear elasticity theory.

Preliminary concepts: Functional and integrity bases, syzygy

Let us review some concepts given, for example, in Zheng (1994), Spencer (1971) and Xiao (1996). Consider a set of isotropic invariants $I_1, \ldots, I_k$ of the tensors $\boldsymbol{C}, \boldsymbol{a} \otimes \boldsymbol{a}$ and $\boldsymbol{b} \otimes \boldsymbol{b}$ (denoted by $\mathcal{S}$).

1. Any single-valued function of $I_1, \ldots, I_B$

$$f(\mathcal{S}) = g(I_1, , \ldots I_B) \tag{8}$$

   is called a *representation* for isotropic scalar-valued functions of $\mathcal{S}$. If one of the invariants in the set $\{I_1, \ldots, I_B\}$ is expressible as a single-valued function of the remainders, the invariant is said to be *functionally reducible*. The representation is said to be *complete*, if any isotropic scalar-valued function of $\mathcal{S}$ can be expressed in the form (8). A functional basis for isotropic scalar-valued functions of $\mathcal{S}$ is the set of invariants in a complete representation for isotropic scalar-valued functions of $\mathcal{S}$. A functional basis is said to be *irreducible*, if none of its proper subsets is a functional basis.
2. If the function $f(\mathcal{S})$ is restricted to polynomial functions, then integrity bases are dealt with. A polynomial invariant is said to be *reducible* if it can be expressed as a *polynomial* in other invariants; otherwise, it is said to be *irreducible*. A set $\mathcal{S}_P$ of polynomial invariants which has the property that any polynomial scalar function can be expressed as a polynomial in members of the given set, is called an *integrity basis*. The integrity basis is said to be *minimal*, if none of its proper subset is an integrity basis. It frequently happens that polynomial relations exist between invariants which do not permit any one invariant to be expressed as a polynomial in the remainder. Such relations are called *syzygies*.
3. An minimal integrity basis is not necessarily an irreducible functional basis, and the later, in general, contains fewer elements than the former.

2. Proof using principal axis invariants that only seven(six) of the ten(nine) invariants are independent

In this paper all subscripts i and j take the values of 1, 2 and 3, unless stated otherwise. If we write

$$\boldsymbol{C} = \sum_{i=1}^{3} \lambda_i^2 \boldsymbol{e}_i \otimes \boldsymbol{e}_i \tag{9}$$

where λ_i and $\boldsymbol{e}_i$, $i = 1, 2, 3$ are the principal values and the principal directions of the right stretch tensor $\boldsymbol{U}$, respectively, and substitute (9) in (4)–(6), we have the expressions:

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \quad I_2 = \lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2, \quad I_3 = (\lambda_1\lambda_2\lambda_3)^2, \tag{10}$$

$$I_4 = \lambda_1^2\zeta_1 + \lambda_2^2\zeta_2 + \lambda_3^2\zeta_3, \quad I_5 = \lambda_1^4\zeta_1 + \lambda_2^4\zeta_2 + \lambda_3^4\zeta_3, \tag{11}$$

$$I_6 = \lambda_1^2\xi_1 + \lambda_2^2\xi_2 + \lambda_3^2\xi_3, \quad I_7 = \lambda_1^4\xi_1 + \lambda_2^4\xi_2 + \lambda_3^4\xi_3, \tag{12}$$

$$I_8 = \sum_{i=1}^{3} \lambda_i^2 \chi_i, \quad I_9 = (\boldsymbol{a} \bullet \boldsymbol{b})^2, \quad I_{10} = \sum_{i=1}^{3} \lambda_i^4 \chi_i, \tag{13}$$

where

$$\zeta_i = (\boldsymbol{a} \bullet \boldsymbol{e}_i)^2, \quad \xi_i = (\boldsymbol{b} \bullet \boldsymbol{e}_i)^2, \quad \chi_i = (\boldsymbol{a} \bullet \boldsymbol{b})(\boldsymbol{a} \bullet \boldsymbol{e}_i)(\boldsymbol{b} \bullet \boldsymbol{e}_i) \quad i = 1, 2, 3. \tag{14}$$

The thirteen terms

$$\lambda_i, \quad \zeta_i, \quad \xi_i, \quad \chi_i \quad (i = 1, 2, 3), \quad \alpha = I_9 = (\boldsymbol{a} \bullet \boldsymbol{b})^2 \tag{15}$$

are invariants with respect to all proper orthogonal tensors $\boldsymbol{Q}$. We note that if we write the strain energy function in the principal axis form, i.e.,

$$W_F = \overline{W}(\lambda_1, \lambda_2, \lambda_3, \boldsymbol{e}_1 \otimes \boldsymbol{e}_1, \boldsymbol{e}_2 \otimes \boldsymbol{e}_2, \boldsymbol{e}_3 \otimes \boldsymbol{e}_3, \boldsymbol{a} \otimes \boldsymbol{a}, \boldsymbol{b} \otimes \boldsymbol{b}), \tag{16}$$