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Biot's problem for a Biot material

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ABSTRACT

The paper presents a solution to the problem of an infinite beam of finite width resting on a poroelastic subgrade. The basic concepts of the elasticity solution are reviewed and the formulation is extended to consider the interaction between the infinite beam and a Biot poroelastic halfspace. A combination of Fourier and Laplace transforms are used to solve the problem. The influence of adhesion and drainage effects is accounted for by considering bounding techniques for prescribing the boundary conditions on the interface. The results of the analytical solution are used to validate the accuracy of a computational approach that uses a standard multi-physics scheme.

1. Introduction

The problem of the flexure of a Bernoulli–Euler beam resting on a deformable medium is a seminal problem that has a wide range of applications in applied mechanics, material science and geomechanics. The deformability of the supporting medium has been approximated by several elementary structural models involving elastic support, ranging from the model consisting of an array of independent spring elements (or piano keys) generally attributed to Winkler (although analogous treatments of the problem can be traced back to the works of Zimmermann, Euler, Bubnov and Hertz (see e.g. Hetényi, 1946; Selvadurai, 1979), to the continuum model (Biot, 1937) with the structural models by Vlazov and Leontiev (1966) and Reissner (1958) occupying an intermediate position. Biot (1937) makes the comment "*A serious objection can be made to the simplifying assumption on which this (Winkler's) elementary theory is based, because it is obvious that the reaction does not depend on the local deflection alone.*" The references to articles covering the topic are quite extensive and no attempt will be made to provide a comprehensive review. The reader is referred to the review articles and texts by Kany (1959), Korenev (1960), Hetényi (1966), Panc (1975), Selvadurai (1979, 2007), Gladwell (1980) and Aleynikov (2011) for advances in the topic. With the Winkler-type elastic support, the problem of the loaded beam on a spring elastic foundation has an elementary solution and an extensive collection of results are given by Hetényi (1946) and Selvadurai (1979). The problem can also be solved in compact form for the case of an elastic continuum where the flexure of the beam and the behavior of the continuum correspond to either a state of *plane strain* or *generalized plane stress*.

The problem is, however, not so straight forward when the beam has a finite width and rests on a complete elastic halfspace. In the treatment of such a problem it is usually assumed that the cross section of the beam imposes a constant displacement

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boundary condition and the beam experiences flexure only in the longitudinal direction. This gives rise to an integral equation that can only be solved in an approximate manner. The approximate solution to this problem was first given by Biot (1937) and this is referred to as *Biot's problem*. Subsequent studies proceeded to improve the method of solution of the governing integral equation, including those by Rvachev (1956a, 1956b, 1958, 1959) and Lekerkerker (1960) who developed techniques for the approximate solution of the governing integral equation. Rakov (1962) and Medovnik, Rakov, and Kh (1970) extended the method by Rvachev (1956b, 1958) to study a three-dimensional array of beams and Protsenko and Sinekop (1973) adopted a variation of Rvachev's method where the displacement across the width of the beam is assumed to be constant with an unknown variation along the longitudinal direction. The arbitrary constant in the deflected shape is evaluated by considering the equilibrium at a cross section along the longitudinal direction. A complete three-dimensional formulation of an infinite strip resting on an isotropic elastic halfspace is given by Protsenko and Rvachev (1976) and accounts for both the longitudinal and transverse flexure of the beam. In such a treatment the Bernoulli–Euler beam is essentially replaced by a Germain–Poisson–Kirchhoff thin plate. Biot's problem also provided a procedure for relating the spring constant of the Winkler model to the elastic constants and other flexural and geometric properties of the beam-elastic halfspace system. Of related interest are the studies related to the *line-contact problem* investigated by Kalker (1972), Sivashinsky (1975), Panek and Kalker (1977) and Tuck and Mei (1983) that examine the narrow rigid die (pizza-cutter) problem, where the contact stresses in the transverse direction approach the problem of the two-dimensional indenter (Sadowsky, 1928) as the length of the indenter increases. Biot's result for the spring constant is derived by matching the maximum flexural moment between the beam with Winkler support with the analogous result for the beam on an elastic halfspace. This gives

$$k_s \approx \frac{1.23\,\mu}{(1-\nu)Cb}\left(\frac{2\mu b^4}{C(1-\nu)EI}\right)^{0.11}$$

where μ and ν are the shear modulus and Poisson's ratio of the elastic halfspace, EI is the flexural rigidity of the beam, b is the half width of the beam and the constant $C \in (1, 1.13)$. Expressions similar to the above were also developed by Vesić (1961a, 1961b) and Barden (1963) by comparing both numerical and experimental results and these are summarized by Selvadurai (1979). An alternative perspective of the interpretation of the Winkler constant was provided by Gibson (1967) who examined the axial surface deformations of an incompressible nonhomogeneous elastic halfspace, where the linear elastic shear modulus varies linearly with depth (i.e. $G(z) = G_0 + mz$). In the particular instance when the shear modulus at the surface becomes zero, the axial surface displacement is discontinuous and the Winkler constant $k = 2m$. Further accounts of developments in this area are given by Selvadurai (1996a, 2007) and Selvadurai and Katebi (2013, 2015). It should be noted that in developments related to the problem of a flexible beam and an elastic halfspace, the contact is assumed to be smooth and bilateral. This places a restriction on the applicability of the developments to situations where localized loading of a flexible beam results in separation and the extent of the zone of separation itself can be an unknown in the problem.

In this paper we first present a brief summary of Biot's problem for an elastic halfspace and indicate a simple procedure that can bound the result for the elasticity problem to take into consideration the influence of adhesive contact between the elastic halfspace and the beam. The work is then extended to consider an infinite beam of finite width resting on a poroelastic halfspace. In the case of the poroelasticity problem, in addition to the question of either smooth or adhesive contact at the beam-elastic medium interface, pore pressure boundary conditions also need to be prescribed. These can range from a completely permeable to a completely impermeable beam-poroelastic medium interface. The paper presents mathematical approaches that yield bounds for the flexural behavior of the infinite beam problem. These bounds are used to assess the accuracy of a computational approach that can be used to examine the interaction of an infinite beam and a poroelastic halfspace.

2. Biot's problem for an elastic halfspace

Prior to considering the problem of an infinite beam on a poroelastic halfspace, it is instructive to outline the basic formulation of the associated elasticity problem. We consider the problem of an infinite beam of finite width resting in *smooth contact* with the surface of an isotropic elastic halfspace (Fig. 1). The beam experiences flexure only in the longitudinal direction and the flexural response of the beam is described by the Bernoulli–Euler classical beam theory. This is not a restriction on the method of analysis and, with suitable modifications the analysis can be extended to thick beam theories that incorporate shear deformation effects.

The flexural behavior of the beam is described by the differential equation

$$EI\frac{d^4 w_b}{dx^4} + \int_{-b}^{b} q(x,y)\,dy = \int_{-b}^{b} p_e(x,y)\,dy \tag{1}$$

The boundary conditions applicable to the problem depend on the contact conditions at the beam-elastic halfspace interface. For convenience of presentation, the region of the surface of the halfspace in contact with the beam is denoted by Γ_c (i.e. $x \in (-\infty,\infty)$; $y \in (-b,b)$; $z=0$) and the combined region of the halfspace exterior to Γ_c is denoted by Γ_e (i.e. $\Gamma_e = \Gamma_{e1} \cup \Gamma_{e2}$ and in $\Gamma_{e1}, x \in (-\infty,\infty)$; $y \in (b,\infty)$; $z = 0$ and in $\Gamma_{e2}, x \in (-\infty,\infty)$; $y \in (-b,-\infty)$; $z=0$). Also we denote $\Gamma_c \cup \Gamma_e = \Gamma$ and $\Gamma_c \cap \Gamma_e = 0$.

(i) For *frictionless contact* between the beam and the elastic halfspace, the following boundary conditions are applicable:

$$u_z(x,y,0) = w_b(x,y); \quad (x,y) \in \Gamma_c \tag{2}$$