International Journal of Engineering Science 91 (2015) 1–11

Contents lists available at ScienceDirect

International Journal of Engineering Science

journal homepage: www.elsevier.com/locate/ijengsci

# Crack opening displacements under remote stress gradient: Derivation with a canonical basis of sixth order tensors

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ARTICLE INFO

*Article history:*
Received 11 September 2014
Received in revised form 22 January 2015
Accepted 30 January 2015
Available online 11 March 2015

*Keywords:*
Cracks
Sixth-order tensor
Anisotropy
Inhomogeneity problem

ABSTRACT

In this paper, we derive the crack opening displacement of a penny-shaped crack embedded in an infinite isotropic elastic medium and subjected to a remote constant stress gradient. The solution is derived by taking advantage of the solution of the equivalent ellipsoidal inclusion problem subjected to a linear polarization. The case of the penny-shaped crack is thereafter investigated by considering the case of a spheroidal cavity which has one principal axis infinitesimally small compared to both others. The derivation of the explicit solution for the inhomogeneity subjected to a remote stress gradient raises the problem of the inversion of a sixth order tensor. For the problem having a symmetry axis (this including the case of penny shaped crack), this problem can be tackled by using a decomposition on the canonical basis for transversely isotropic sixth order tensors.



## 1. Introduction

This paper provides the explicit solution for the penny shaped crack opening displacement subjected to remote stress gradient. The problem of a crack in an infinite elastic body has been studied for a long time; there is an abondant literature on this subject which has been summarized in the classic books of (Mura, 1987; Nemat-Nasser & Hori, 1999; Kachanov, Shafiro, & Tsukrov, 2003). Classically, the problem of crack subjected to an applied remote strain (or equivalently stress) can be addressed using the Eshelby (1957, 1959) formalism. Eshelby's solution for inclusions and for equivalent inhomogeneity problems are fundamental to many problems in material science, mechanics of composite, etc. In the terminology of Eshelby (1957) and Mura (1987), an inclusion denotes a subdomain subjected to an eigenstrain while an inhomogeneity is a subdomain whose elastic properties differ from those of the surrounding medium. The main Eshelby's result is well known for the case of a prescribed constant eigenstrain: it shows that a constant strain field is generated inside an ellipsoidal inclusion (Eshelby, 1957) while the exterior point solution (outside the inclusion) is heterogeneous (Eshelby, 1959). The Eshelby's equivalent method handles the problem of a single ellipsoidal inhomogeneity by replacing it with an inclusion having properly chosen eigenstrains. The results for the penny shaped cracks are recovered when two semi-axes of the ellipsoidal inhomogeneity are equal and the last is infinitesimally small compared to the others.

Due to its simplicity, the Eshelby's solution is the basis of numerous ways to understand the behavior of heteregeneous and cracked materials, and finally was used extensively to construct numerous constitutive equations of these materials. A few recent references show that this field of research is still active for studying cracked materials (Monchiet, Gruescu, Cazacu, & Kondo, 2012; Mihai & Jefferson, 2011). However, some works have shown also that taking into account the

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http://dx.doi.org/10.1016/j.ijengsci.2015.01.002
0020-7225/

influence of strain (or stress) gradients is of fundamental significance when studying cracked materials, leading to the use of constitutive equations of gradient elasticity type (Mühlich, Zybell, Hütter, & Kuna, 2013; Gourgiotis & Georgiadis, 2009). So, extending the Eshelby's formalism to inclusions and cracks subjected to remote strain (or stress) gradients allows to provide a new fundamental background to the understanding of heterogeneous materials. Such a solution was provided in Monchiet and Bonnet (2013) for the case of an spheroidal inclusion subjected to a remote gradient, based on older fundamental results (see also Monchiet and Bonnet (2011) for the case of a spherical inclusion). Indeed, following Eshelby's work, Sendeckyj (1967), Moschovidis (1975), Asaro and Barnett (1975), generalize Eshelby's solution to the case of prescribed polynomial fields. In these studies, the following result has been proved: *the strain in an ellipsoidal subdomain of an infinite linear elastic medium which undergoes an eigenstrain on the form of a polynomial of degree N, is also a polynomial with the same degree N*. The expansion of the eigenstrain and the interior point solution for the strain field along polynomial functions introduce tensors of order $2, 3, 4$ etc and higher order Eshelby tensors of order $6, 8$ etc. It has been pointed out in Mura (1987) that the strain disturbance due to a polynomial type remote strain field of degree $N$ can be simulated by an appropriate polynomial eigenstrain field of degree $N$. If the inclusion problem can be traced back to these works, the related heterogeneity problem requires the inversion of high order tensors. For example, for a remote strain which is linear with coordinates, the problem involves the inversion of a sixth order tensor. Although this inversion can be performed numerically, it is obviously of interest to derive closed form solutions which can cover many applications, as performed in Monchiet and Bonnet (2013).

In the present paper, we use a canonical basis of transversely isotropic sixth order tensors (Monchiet & Bonnet, 2013) which has the advantage to reduce drastically the size of the system for the case of a spheroidal inhomogeneity (ellipsoidal inhomogeneity with symmetry axis). It allows to provide an easy inversion of high order tensors and therefore to find an easy way to solve the heterogeneity problem. Thus, the linear system which is initially of dimension 18 reduces to five independent systems of dimension $1, 1, 2, 3$ and 4. When the case of a void is considered the problem exhibits some impotent strains and degenerates into five independent systems of dimension $1, 1, 2, 2$ and 3. Finally, the case of a penny shaped crack is considered by expanding the solution with respect to the aspect ratio of the spheroidal inhomogeneity.

## 2. Inhomogeneity problem with applied remote strain or stress gradient

Consider an ellipsoidal inhomogeneity embedded in an infinite elastic matrix and centered at the origin of the cartesian frame $(x_1, x_2, x_3)$. Let us denote by $a_1, a_2, a_3$ the semi-axes of the ellipsoid along the three axes of the cartesian frame. The subdomain $\Omega$ of the inhomogeneity is defined by:

$$U(x) = \eta_{ij} x_i x_j - 1 \leqslant 0 \quad (1)$$

with:

$$\eta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1/a_i^2 & \text{if } i = j \end{cases} \quad (2)$$

Both the inhomogeneity and the infinite domain are assumed to be isotropic and we denote by $\mu, \lambda$ and $\mu_0, \lambda_0$ their respective Lamé coefficients. The inhomogeneity is subjected to a remote strain field taken on the form $\varepsilon_{ij}(x) = \varepsilon^{\infty}_{ijk} x_k$ or equivalently to the remote stress field $\sigma^{\infty}_{ij}(x) = \sigma^{\infty}_{ijk} x_k$ where $\varepsilon^{\infty}_{ijk}$ and $\sigma^{\infty}_{ijk}$ are constant ($\underline{x}$-independent). This problem is called "second order heterogeneity problem" since a uniform gradient of strain (or stress) is applied instead of the constant strain considered by Eshelby in the first order problem. The strain and stress are related outside the heterogeneity by:

$$\sigma^{\infty}_{ijk} = C^0_{ijpq} \varepsilon^{\infty}_{pqk} \quad (3)$$

At infinity, the equilibrium for the applied remote stress is:

$$\sigma^{\infty}_{ijj} = C^0_{ijpq} \varepsilon^{\infty}_{pqj} = 0 \quad (4)$$

$\varepsilon^{\infty}_{ijk}$ and $\sigma^{\infty}_{ijk}$ being symmetric with respect to indices $i$ and $j$ and accounting for the equilibrium condition (4), we deduce that they both depend on 15 independent coefficients.

The strain disturbance due to the application of $\varepsilon^{\infty}_{ijk}$ at infinity can be obtained by considering the following appropriate inclusion problem: an ellipsoidal domain, defined by (1), is subjected to a prescribed eigenstrain of the form $\varepsilon^{*}_{ij}(x) = e_{ijk} x_k$ (with $e_{ijk} = e_{jik}$). As shown in Sendeckyj (1967), Moschovidis (1975), Moschovidis and Mura (1975) and also in Asaro and Barnett (1975) for the more general case of an anisotropic matrix, the solution is defined by a linear strain within the inclusion:

$$\varepsilon_{ij}(x) = S_{ijkpqr} e_{pqr} x_k \quad (5)$$

where $S_{ijkpqr}$ are the components of a sixth-order Eshelby tensor (to make reference to the common fourth order Eshelby tensor when dealing with constant eigenstrain within the inclusion domain). The expression of the stress components (still within the inclusion) are:

$$\sigma_{ij}(x) = C^0_{ijmn} (S_{mnkpqr} e_{pqr} - e_{mnk}) x_k \quad (6)$$