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A geometrically nonlinear beam model based on the second strain gradient theory

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ABSTRACT

The geometrically nonlinear governing differential equation of motion and corresponding boundary conditions of small-scale Euler–Bernoulli beams are achieved using the second strain gradient theory. This theory is a non-classical continuum theory capable of capturing the size effects. The appearance of many higher-order material constants in the formulation can certify that it appropriately assesses the behavior of extremely small-scale structures. A hinged–hinged beam is chosen as an example to lay out the nonlinear size-dependent static bending and free vibration behaviors of the derived formulation. The results of the new model are compared with the previously obtained results based on the strain gradient theory and the classical theory.

1. Introduction

Nowadays, the usage of beam-shaped structures has widely spread in micro- and nano-electromechanical systems (MEMS and NEMS) as elements such shock sensors (Currano, Yu, & Balachandran, 2010), micro-actuators (Li & Chew, 2014), accelerometers (Davies, George, Gower, & Holmes, 2014), bio-MEMS (Djurić, Jokić, & Peleš, 2014), atomic force microscopes, and so on Kahrobaiyan, Rahaeifard, and Ahmadian (2011). Consequently, scholars are extremely interested in precise modeling of static and dynamic behaviors of them. As the beams utilized in MEMS and NEMS structures have thicknesses in the order of microns and sub-microns, the experimentally validated small scale effects would be considerable in their behavior (Fleck, Muller, Ashby, & Hutchinson, 1994; Andrew & Jonathan, 2005; Chan, Fu, & Lu, 2011; Ran, Fu, & Chan, 2013). In fact, experiments manifest that the size-dependent behavior is an intrinsic feature of materials when they are used in small-scale structures. When the characteristic size of a beam, such as its thickness or diameter, is in the order of the internal material length scale parameters, size-dependent behavior will arise for it (Kong, Zhou, Nie, & Wang, 2008). In the classical continuum mechanics, there exists no material length scale parameter. Thus, the classical theory is unable to capture the size-dependency of small-scale structures. Therefore, some non-classical continuum theories such as the couple stress theory and higher order gradient theories have been proposed to validly predict the size-dependent mechanical behavior of small-scale structures. These theories usually consider some higher-order stresses besides the classical stress, and there are material length scale parameters in addition to the classical material constants in the corresponding constitutive relations.

The couple stress theory, as a non-classical continuum theory, was proposed in 1960s (Mindlin & Tiersten, 1962; Toupin, 1962; Koiter, 1963). In this theory, there exist couple stress components, as higher order stresses. There are two higher order

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material constants besides the two well-known classical Lame ones in the constitutive equations of the theory which make it capable to predict the size-dependent behaviors. Along this line of thought, Asghari, Kahrobaiyan, Rahaeifard, and Ahmadian, 2011a, 2011b presented the problem of size-effects in Timoshenko beams. Moreover, Su and Liu (2014) used this theory to study size-effects in the free vibration behaviors of periodic cellular solids. In 2002, Yang et al. suggested the so-called modified couple stress theory in which exists only one higher-order material constant (Yang, Chong, Lam, & Tong, 2002). Asghari and Kahrobaiyan used this theory to investigate the size effects in some problems (Asghari et al., 2011a, Asghari, Rahaeifard, Kahrobaiyan, & Ahmadian, 2011b; Kahrobaiyan, Asghari, & Ahmadian, 2014). Also, Fu and Zhang (2010) studied the size-dependent buckling behavior of micro-tubules, and Mohammad Abadi and Daneshmehr (2014) proposed buckling analysis of composite laminated beams in the context of this theory. In addition, some scholars such as Xia, Wang, and Yin (2010) provided studies based on nonlinear models. In fact, these nonlinear studies are very noteworthy because in beams with two immovable supports which are used in small-scale structures, nonlinearity is a common phenomenon due to the mid-plane stretching, and causes the results to be changed immensely. In view of this fact, Tang, Ni, Wang, Luo, and Wang (2014) studied three dimensional vibration behavior of curved microtubes, and proposed a nonlinear model for it based on modified couple stress theory, and Asghari, Kahrobaiyan, and Ahmadian (2010) presented a nonlinear Timoshenko beam model in the framework of this theory.

In 1965, Mindlin developed a higher-order gradient theory with many higher-order material constants which make it highly capable to predict small-scale effects (Mindlin, 1965). He considered derivatives of the strain tensor up to the second order in the strain energy density, in which sixteen higher-order material constants exist. This theory is called the second strain gradient theory. In 1968, he focused in the special case of the theory in which the dependency on the second gradients was neglected. In these conditions, the (first) strain gradient theory is obtained with five higher-order material constants in the corresponding constitutive equations. Ansari, Gholami, Faghih Shojaei, Mohammadi, and Sahmani (2013) utilized this theory to study the bending, buckling and free vibration responses of Timoshenko micro-beams made of functionally graded materials (FGMs).

Lam, Yang, Chong, Wang, and Tong (2003) suggested a version of the strain gradient theory with three higher-order material constants. In the framework of this theory, some researches have investigated the static and dynamic behaviors of linear Euler–Bernoulli and Timoshenko beams (Kong, Zhou, Nie, & Wang, 2009; Wang, Zhao, & Zhou, 2010; Li, Feng, & Cai, 2014; Mousavi, Paavolaa, & Baroudi, 2014). Also, Vatankhah et al. presented the problem of boundary control of vibrating strain gradient micro-scale beams (Vatankhah, Kahrobaiyan, Alasty, & Ahmadian, 2013; Vatankhah, Najafi, Salarieh, & Alasty, 2013; Vatankhah, Najafi, Salarieh, & Alasty, 2014a). Moreover, they employed the strain gradient theory to investigate the size-dependent nonlinear forced vibration, and exact controllability problem of a vibrating non-classical Euler–Bernoulli micro-beam (Vatankhah, Kahrobaiyan et al., 2013; Vatankhah, Najafi et al., 2013; Vatankhah, Najafi, Salarieh, & Alasty, 2014b). As mentioned before, nonlinear analyses are very crucial to be noted, and many researches considered nonlinearity in their studies. For example, Asghari et al. considered nonlinearity in some problems of static and dynamic behaviors of micro-beams based on strain gradient theory (Kahrobaiyan, Asghari, Rahaeifard, & Ahmadian, 2011; Kahrobaiyan et al., 2011; Asghari, Kahrobaiyan, Nikfar, & Ahmadian, 2012; Kahrobaiyan, Rahaeifard, Tajalli, & Ahmadian, 2012). Moreover, Lazopoulos, Lazopoulos, and Palassopoulos (2014) studied the problem of nonlinear bending of strain gradient elastic Euler–Bernoulli beams, and Ghayesh, Amabili, and Farokhi (2013) employed the strain gradient theory to investigate the nonlinear forced vibrations of micro-beams.

Lazar, Maugin, and Aifantis (2006) have proposed a simplified but straightforward version of the second strain gradient theory to study a screw dislocation and an edge dislocation. Some problems of surface effects, dislocations, and disclinations using this version of the second strain gradient theory have been studied by some scholars (Deng, Liu, & Liang, 2007; Polizzotto, 2014). In 2012, Shodja, Ahmadpoor, and Tehranchi (2012) provided analytical formulation of material length scale parameters associated with the original second strain gradient theory for face centered cubic (fcc) materials, and assessed the linear size-dependent static behavior of a cantilever beam as a structural case (Shodja et al., 2012).

In this paper, a geometrically nonlinear size-dependent beam formulation is developed in the framework of the second strain theory. As a special case, the practical case of hinged–hinged beams is considered and static response of them in a problem is investigated. Then, the free vibration of the beams is analytically treated and the obtained results are compared with those evaluated based on the previously available formulations including linear and nonlinear strain gradient based theories as well as linear and nonlinear models based on the classical theory.

2. Derivation of the governing equation and associated boundary conditions

Based on the second strain gradient theory introduced by Mindlin (1965), the strain energy density $\bar{u}$ for linear isotropic elastic materials is written as

$$\begin{aligned}\bar{u} = &\frac{1}{2}\lambda\varepsilon_{ii}\varepsilon_{jj} + \mu\varepsilon_{ij}\varepsilon_{ij} + a_1\eta_{ijj}\eta_{ikk} + a_2\eta_{iik}\eta_{kjj} + a_3\eta_{iik}\eta_{jjk} + a_4\eta_{ijk}\eta_{ijk} + a_5\eta_{ijk}\eta_{kji} + b_1\xi_{iijj}\xi_{kkll} + b_2\xi_{ijkk}\xi_{ijll} + b_3\xi_{iijk}\xi_{jkll} \\ &+ b_4\xi_{iijk}\xi_{llkj} + b_5\xi_{iijk}\xi_{lljk} + b_6\xi_{ijkl}\xi_{ijkl} + b_7\xi_{ijkl}\xi_{jkli} + c_1\varepsilon_{ii}\xi_{jjkk} + c_2\varepsilon_{ij}\xi_{ijkk} + c_3\varepsilon_{ij}\xi_{kkij} + b_0\xi_{iijj},\end{aligned} \tag{1}$$

in which