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Analytical solutions to the axisymmetric elasticity and thermoelasticity problems for an arbitrarily inhomogeneous layer

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A B S T R A C T

In this paper, we present a technique for constructing an analytical solution to the axisymmetric elasticity and thermoelasticity problems in terms of stresses for an inhomogeneous layer, whose elastic and thermophysical properties vary arbitrarily within the thickness-coordinate. By making use of the direct integration method, the equilibrium and compatibility equations are reduced to the governing Volterra-type integral equations accompanied with both integral and local boundary conditions for the key functions. To solve the obtained governing equations, we employed the resolvent-kernel technique which results in closed-form analytical expressions for the key functions. Having determined the key functions, the stress-tensor components are found through the relationship established by the integration of equilibrium equations. The same solution procedure is employed for solving the steady-state heat-conduction problem in an inhomogeneous layer. Typical numerical examples are discussed.

1. Introduction

The elastic response of inhomogeneous solids to force and thermal impacts is one of the most discussed subjects in the field of linear elasticity and thermoelasticity in recent decades. Under assumption that the elastic modulii can be functions of the coordinates, the linear Hooke's (or Duhamel's, for the thermoelastic case) law is usually accepted as a basic model for such a response on a macroscopic scale. Under this assumption, the governing equations for the relevant problems of elasticity and thermoelasticity appear to possess variable coefficients and allow for acquisition of their explicit analytical solutions in very limited cases of inhomogeneity (Tanigawa, 1995). Obviously, analysis of thermal and mechanical performance of arbitrarily-inhomogeneous solids presents a challenge for both analytical and numerical modes of attack due to significant mathematical complications and calls for a number of simplifying assumptions, which are usually accepted in the relevant literature but are potentially insufficient.

From a historical prospective, the pioneering works were focused on one- and two-dimensional elasticity problems with inhomogeneous materials as they were primarily concerned with problems of geophysics, such as the propagation of elastic waves in seismic processes or stress distributions in soil due to the steady-state local pressure caused by building structures, etc. For adequate analysis of these problems, it was important to account for the variation in soil properties with respect to

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depth. A number of approximate solutions for certain problems in elasticity theory for specific cases of inhomogeneity have been developed, e.g., by Winkler (1867), Aichi (1922), Sezawa (1931), Wilson (1942), Ewing, Jardetzki, and Press (1957), Gibson (1967), Kassir (1970), Awojobi and Gibson (1973), Muravskii (2001) and many others.

In the second half of the twentieth century, the advancement in technologies that implemented composite materials by combining dissimilar constituents inspired a surge of interest in the analysis of inhomogeneous structures (see, e.g., Hashin, 1964; Olszak, 1959), which lead to the establishment of basic methods in this area. In the 1980s, new technologies for the fabrication of functionally-graded materials (this term has been suggested in Japan during the space-exploration program, see, e.g., Koizumi, 1997; Miyamoto, Niino, & Koizumi, 1997; Rabin & Shiota, 1995) were widely developed. The basic purpose for employment of the functionally-graded materials was to create an intermediate layer between the contrasting materials (e.g., metal and ceramics) in order to eliminate or control the residual stresses and thermal deformations (see Wetherhold, Seelman, & Wang, 1996) caused by the mismatch of material properties. The exhaustive reviews on the subject of functionally-graded materials were published by Ilschner (1997), Rabin and Shiota (1995), Rödel and Neubrand (1997) and Birman and Byrd (2007). The widespread interest in continuously inhomogeneous materials has developed a number of analytical and numerical methods, some of which have become dominant in scientific literature.

One of these analytical methods is based on the construction of exact solutions to the elasticity and thermoelasticity problems for the solids, whose material properties are assumed to be given by elementary functions (i.e., linear, power, exponential, etc.) of one of the spatial coordinates. The number of papers devoted to the analysis of these particular cases of inhomogeneity is quite large and grows rapidly, which makes it nearly impossible to present the extensive reviews of the relevant references (see, e.g., Muravskii, 2001; Tokovyy & Ma, 2009; Wang, Tzeng, Pan, & Liao, 2003). The popularity of this approach was provoked by the possibility of obtaining a closed-form analytical solution by means of the classical methods of mathematical physics. On the other hand, the application of these solutions is limited due to the very specific cases of inhomogeneity which can be attempted by this means.

An alternative approach is based on the representation of arbitrarily-inhomogeneous solids by assembling perfectly connected homogeneous layers (Liew, Kitipornchai, Zhang, & Lim, 2003; Liu, Ke, Wang, Yang, & Alam, 2012; Zhang & Hasebe, 1999). This method is known as the discrete-layer approach (Ramirez, Heyliger, & Pan, 2006). Having solved the problem for each homogeneous layer, the solutions then are tailored by making use of the interface conditions to obtain the solution for entire solid satisfying the original boundary conditions on its surfaces. A weakness of this approach is that there are possible stress discontinuities at interfaces and weak convergence of the constructed solution with increment of the number of layers (Watremetz, Baietto-Dubourg, & Lubrecht, 2007). To overcome these difficulties, a combination of two foregoing approaches can be employed when the material properties of each sub-layer are assumed to be elementary functions (for instance, linear Plevako, 2002 or exponential Guo & Noda, 2007 ones, etc.) so that they can be managed as continuous at the interfaces.

A method for the solution of plane elasticity problems for inhomogeneous solids by means of the complex variable technique has been suggested by Mishiku and Teodosiu (1966). According to this approach, the original problems are reduced to the conjugation problems, which, in turn, are solved by means of successive approximations.

There are a number of numerical (Meguid & Zhu, 1995; Reddy & Cheng, 2001; Tarn, Wang, & Wang, 1996) and analytic-numerical (Kushnir, Popovych, & Harmatii, 2001; Kushnir, Popovych, & Vovk, 2008) methods for analysis of elastic behavior of inhomogeneous solids.

An efficient approach to the analysis of elastic and thermoelastic response of arbitrarily-inhomogeneous solids is based on the reduction of the original problems to solutions of integral equations. Theoretical background for this approach has been substantiated by Lopatinskii (1953) and Fichera (1961). This method proved its efficiency for a number of one-dimensional problems (Clements & Rogers, 1978; Furuhashi & Kataoka, 1967; Li, Peng, & Lee, 2010; Panferov & Leonova, 1975). Based on the method of direct integration (Tokovyy, 2014), Vihak (Vigak) and his followers (see Tokovyy, Kalynyak, & Ma, 2014) systemized this approach by suggesting a clear algorithm, which allowed for the reduction of the original problems to the Volterra-type integral equations of second kind with accompanying boundary and integral conditions. This algorithm along with the application of the resolvent-kernel technique allowed for the construction of solutions to the formulated problems in explicit analytical form, which can be used for numerical implementation as well as for further analytical treatment (Tokovyy & Ma, 2013).

In this paper, we utilize the direct integration method for the case of axisymmetric elasticity and thermoelasticity problems in terms of stresses for an infinite layer, whose elastic modulii are arbitrary functions of the transversal coordinate. The original equilibrium and compatibility equations are reduced to two governing equations for the key functions with corresponding boundary conditions. By making use of the Hankel integral transformation, we separate the variables in the obtained equations and then reduce them to the solution of Volterra's integral equation of second kind. By solving the latter equation with application of the resolvent-kernel, the key functions are determined and then the stress-tensor components are constructed by means of the relations, obtained on the basis of the equilibrium equations. The same strategy is employed for the solution of the axisymmetric heat-conduction problem in the considered domain.

2. Formulation of the problem

Consider an axisymmetric problem of elasticity and thermoelasticity for a layer $0 \leqslant \rho < \infty,\ |z| \leqslant h$, whose material properties, i.e., the Young's modulus, E, the Poisson's ratio, ν, and the linear thermal expansion coefficient, α, are arbitrary func-