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Dispersion of Rayleigh waves in weakly anisotropic media with vertically-inhomogeneous initial stress

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ABSTRACT

Herein we present a procedure by which a high-frequency asymptotic formula can be derived for dispersion relations of Rayleigh waves that propagate in various directions along the free surface of a vertically-inhomogeneous, prestressed, and generally anisotropic half-space. The procedure is based on three assumptions, namely: (i) the incremental elasticity tensor of the material half-space can be written as the sum of a homogeneous isotropic part $\mathbb{C}^{\mathrm{Iso}}$ and a depth-dependent perturbative part $\mathbb{A}$; (ii) at the free surface both the initial stress and $\mathbb{A}$ are small as compared with $\mathbb{C}^{\mathrm{Iso}}$; (iii) the mass density, the initial stress, and $\mathbb{A}$ are smooth functions of depth from the free surface. We derive formulas and Lyapunov-type equations that can iteratively deliver each term of an asymptotic expansion of the surface impedance matrix, which leads to the aforementioned high-frequency asymptotic formula for Rayleigh-wave dispersion. As illustration we consider a thick-plate sample of AA 7075-T651 aluminum alloy, which has one face treated by low plasticity burnishing that induced a (depth-dependent) prestress at and immediately beneath the treated surface. We model the sample as a prestressed, weakly-textured orthorhombic aggregate of cubic crystallites and work out explicitly, up to the third order, the dispersion relations that pertain to Rayleigh waves propagating in several directions along the treated face of the sample.

1. Introduction

Recently Man, Nakamura, Tanuma, and Wang (2015) developed a general procedure, under the framework of linear elasticity with initial stress (Biot, 1965; Hoger, 1986; Man & Carlson, 1994; Man & Lu, 1987), for obtaining a high-frequency asymptotic formula for the dispersion of Rayleigh waves propagating in a vertically-inhomogeneous, prestressed and anisotropic medium. That work was meant to serve as the mathematical foundation for a nondestructive measurement technique to monitor the retention of protective surface and subsurface compressive stresses which are put in metal parts (e.g., critical components of aircraft engines) by surface treatments for fatigue-life enhancement. The theory in Man et al. (2015) does not consider the effects of surface roughness on Rayleigh-wave dispersion; it covers only surface treatments (e.g., low plasticity burnishing (LPB), which leaves a mirror-smooth surface finish) where such effects can be ignored. On the other hand, that theory is developed with the constitutive equation in linear elasticity with initial stress put in its most general form, which makes derivation of explicit dispersion relations difficult.

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Here we adapt the general procedure in Man et al. (2015) to the case where the incremental elasticity tensor $\mathbb{L}$ can be written as the sum of an isotropic part $\mathbb{C}^{\rm Iso}$ and a perturbative part $\mathbb{A}$. Under a Cartesian coordinate system where the material medium occupies the half-space $x_3 \leqslant 0$, the perturbative part $\mathbb{A}(\cdot)$, the initial stress $\mathring{\boldsymbol{T}}(\cdot)$, and the mass density $\rho(\cdot)$ are assumed to be smooth functions of x_3. Moreover, at the free surface $x_3 = 0$ of the material medium $\mathbb{A}(0)$ and $\mathring{\boldsymbol{T}}(0)$ are assumed to be sufficiently small as compared with $\mathbb{C}^{\rm Iso}$ that, for all expressions and formulas which depend on $\mathbb{A}(0)$ and $\mathring{\boldsymbol{T}}(0)$, it suffices to keep only those terms linear in the components of these tensors. Under this setting, after outlining some preliminaries in Section 2, we derive in Sections 3–5 specific formulas with which the procedure presented in Man et al. (2015) can be implemented to solve the direct problem of deriving high-frequency asymptotic formulas for dispersion relations that pertain to Rayleigh waves with various propagation directions. Once dispersion curves can be generated when requisite data on material and stress are given, the inverse problem of inferring stress retention from Rayleigh-wave dispersion can be attacked by an iterative approach in further studies.

In Section 6, we present an illustrative example where we derive Rayleigh-wave dispersion relations for a thick-plate sample of an AA 7075-T651 aluminum alloy that carries a prestress induced by prior LPB-treatment. The sample is modeled as a weakly-textured orthorhombic aggregate of cubic crystallites. The prestress in the sample was ascertained by destructive means (X-ray diffraction and hole-drilling), and so were the relevant texture coefficients (X-ray diffraction). To shed light on how crystallographic texture would affect the dispersion relations, we prescribe two other textures to the sample and repeat the calculations with the prestress and material parameters unchanged.

2. Preliminaries

In a Cartesian coordinate system let (x_1, x_2, x_3) be the Cartesian coordinates of place $\mathbf{x}$, and let $\boldsymbol{u} = \boldsymbol{u}(\mathbf{x}) = (u_1, u_2, u_3)$ be the displacement at $\mathbf{x}$ pertaining to the superimposed small elastic motion. We work in the theoretical context of linear elasticity with initial stress, under which the constitutive equation can be put in the form (cf. Man & Carlson (1994), Man & Lu (1987))

$$\boldsymbol{S} = \mathring{\boldsymbol{T}} + \boldsymbol{H}\mathring{\boldsymbol{T}} + \mathbb{L}[\boldsymbol{E}]; \tag{1}$$

here $\boldsymbol{S} = (S_{ij})$ is the first Piola–Kirchhoff stress, $\mathring{\boldsymbol{T}} = (\mathring{T}_{ij})$ the initial stress, $\boldsymbol{H} = (\partial u_i/\partial x_j)$ the displacement gradient pertaining to the superimposed small elastic motion, and $\boldsymbol{E} = (\boldsymbol{H} + \boldsymbol{H}^T)/2$ the corresponding infinitesimal strain, where the superscript T denotes transposition; $\mathbb{L}$ is the incremental elasticity tensor which, when regarded as a fourth-order tensor on symmetric tensors, has its components L_{ijkl} $(i, j, k, l = 1, 2, 3)$ satisfying the major and minor symmetries.

We choose the Cartesian coordinate system so that the material half-space occupies the region $x_3 \leqslant 0$ whereas the 1- and 2-axes are chosen arbitrarily. In this paper we assume that the initial stress $\mathring{\boldsymbol{T}} = \mathring{\boldsymbol{T}}(x_3)$, the incremental elasticity tensor $\mathbb{L} = \mathbb{L}(x_3)$, and the mass density $\rho = \rho(x_3)$ are smooth functions of the coordinate x_3 $(x_3 \leqslant 0)$. Here and hereafter we use the term "smooth function" to denote an infinitely differentiable function all of whose derivatives are bounded and continuous. We assume that the initial stress $\mathring{\boldsymbol{T}}$ satisfies the equation of equilibrium div $\mathring{\boldsymbol{T}} = \mathbf{0}$, and that the surface $x_3 = 0$ of the half-space is free of traction, which implies that the components $\mathring{T}_{i3}(x_3)$ $(i = 1, 2, 3)$ of $\mathring{\boldsymbol{T}}$ vanish at the surface $x_3 = 0$. We call $-x_3 \geqslant 0$ the depth of place $\mathbf{x}$ beneath the free surface $x_3 = 0$.

In what follows we suppose that $\mathbb{L}$ can be written as a sum of two terms: a principal part $\mathbb{C}^{\rm Iso}$ which is *homogeneous* and isotropic, and a perturbative part $\mathbb{A} = \mathbb{A}(x_3)$ which is a smooth function of x_3 $(x_3 \leqslant 0)$ and is generally anisotropic. Then $\mathbb{L}$ can be expressed as a fourth-order tensor on symmetric tensors $\boldsymbol{E}$ in the form

$$\mathbb{L}[\boldsymbol{E}] = \mathbb{C}^{\rm Iso}[\boldsymbol{E}] + \mathbb{A}[\boldsymbol{E}] = \lambda(\mathrm{tr}\,\boldsymbol{E})\mathbf{I} + 2\mu\boldsymbol{E} + \mathbb{A}[\boldsymbol{E}], \tag{2}$$

where $\mathbf{I}$ is the identity matrix, λ and μ are the Lamé constants that pertain to $\mathbb{C}^{\rm Iso}$, and $\mathbb{A}$ can be written under the Voigt notation as a 6×6 symmetric matrix $(a_{rs}(x_3))$ with its components a_{rs} being smooth functions of x_3 $(x_3 \leqslant 0)$. In the present study we adopt the following basic assumption:

$(*)$ At the free surface $x_3 = 0$, the perturbative part $\mathbb{A}$ of $\mathbb{L}$ and the initial stress $\mathring{\boldsymbol{T}}$ are sufficiently small as compared with the isotropic part $\mathbb{C}^{\rm Iso}$ of $\mathbb{L}$ (i.e., $\|\mathring{\boldsymbol{T}}(0)\| \ll \|\mathbb{C}^{\rm Iso}\|, \|\mathbb{A}(0)\| \ll \|\mathbb{C}^{\rm Iso}\|$, where $\|\cdot\|$ denotes the Euclidean norm) that for all expressions and formulas which depend on $\mathbb{A}(0)$ and $\mathring{\boldsymbol{T}}(0)$ it suffices to keep only those terms linear in the components of these tensors.

Throughout this paper, we do not put any condition on the x_3-derivatives of $\mathbb{A}(x_3)$ and of $\mathring{\boldsymbol{T}}(x_3)$ at $x_3 = 0$.

Substituting the componentwise expression of (1) into the equations of motion with zero body force, we obtain elastic wave equations of the form

$$\rho\frac{\partial^2}{\partial t^2}u_i = \sum_{j,k,l=1}^{3}\frac{\partial}{\partial x_j}\left(B_{ijkl}\frac{\partial u_k}{\partial x_l}\right), \quad i = 1, 2, 3, \tag{3}$$