A solid–fluid mixture theory of porous media

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A B S T R A C T

The theories of mixtures in the framework of continuum mechanics have been developed throughout the sixties and seventies. In this paper, we shall focus on one of the typical constitutive theories obtained by Bowen (1976) for mixture of elastic materials. We employ a different procedure in the exploitation of the entropy principle, by the use of Lagrange multipliers for a solid–fluid mixture. It is interesting to note that these two theories are completely equivalent concerning the constitutive restrictions.

The general results are then applied to the studies of porous media by introducing the concept of porosity. The resulting theory is consistent with Darcy's law and the concepts of pore fluid pressure and effective stress principle in soil mechanics.

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1. Introduction

The theories of mixtures in the framework of rational continuum thermodynamics have been developed throughout the sixties and seventies, and subsequent development in various constitutive theories and thermodynamic analysis are too numerous to document (for related references, see Rajagopal & Tao, 1995). Unlike Irreversible Thermodynamics which postulates a *generalized* Gibbs relation and mandates *linear* constitutive relations in thermodynamic *fluxes/forces*, Rational Thermodynamics relies on the entropy principle to exploit general restrictive conditions on constitutive functions, linear or non-linear. There are basically two different approaches in exploiting the entropy principle. One is the so-called Coleman–Noll procedure, by eliminating the entropy supply with the energy supply of the energy equation. While the other is the so-called Müller–Liu procedure by employing the Lagrange multipliers. Generally speaking, the Coleman–Noll procedure is more restrictive because it requires some specific assumptions concerning the relation between the entropy flux/supply and energy flux/supply. However, it is more widely used owing to its relatively simpler mathematical calculations. For mixture theories, Müller was the first to employ Lagrange multipliers for mixtures of fluids (Müller, 1973) and later many similar theories have been presented mostly with linear constitutive equations to facilitate the tedious calculations and to avoid possible mathematical difficulties.

One of the typical constitutive theory of elastic solid–fluid mixture was presented by Bowen (1976), employing Coleman–Noll procedure for large deformation without linear constitutive assumptions. In this paper, we shall employ the Lagrange multipliers in analyzing the entropy principle for such a constitutive theory in order to compare with the general results obtained by Bowen. In doing so, we refrain from making isotropic and linear constitutive assumptions, but to facilitate mathematical difficulties in dealing with Lagrange multipliers, some simplifying assumptions are made, for one, the Lagrange

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multiplier for energy equation is identified with the reciprocal of the absolute temperature. It is found that these two theories are completely equivalent concerning the constitutive restrictions.

After a brief review of essential features of the theories of mixtures and the exploitation of constitutive restriction for an elastic solid–fluid mixture with Lagrange multipliers, the general results are applied to the formulation of a theory of porous media by introducing the concept of porosity. The resulting theory is consistent with Darcy's law and the concepts of pore fluid pressure and effective stress principle in soil mechanics.

2. Theories of mixtures

We shall briefly review some essential features for continuum framework of mixture theories. We consider a mixture of N constituents, all of which are supposed to be able to occupy the same region of space simultaneously. Let $\mathcal{B}_\alpha$ denote the αth constituent and κ_α be its reference configuration and denote $B_\alpha = \kappa_\alpha(\mathcal{B}_\alpha)$. The motion of $\mathcal{B}_\alpha$ is a smooth mapping,

$$\chi_\alpha : B_\alpha \times \mathbb{R} \to \mathbb{E}, \quad \boldsymbol{x} = \chi_\alpha(\boldsymbol{X}_\alpha, t), \quad \boldsymbol{X}_\alpha \in B_\alpha$$

for each constituent $\alpha = 1, \ldots, N$. It states that for different motion of each constituent, at the instant t, there is a material point $X_\alpha \in \mathcal{B}_\alpha$ in each constituent, $\boldsymbol{X}_\alpha = \kappa_\alpha(X_\alpha)$, that occupies the same spatial position $\boldsymbol{x}$ in the Euclidean space $\mathbb{E}$. The velocity and the deformation gradient of each constituent are defined as

$$\boldsymbol{v}_\alpha = \frac{\partial}{\partial t} \chi_\alpha(\boldsymbol{X}_\alpha, t), \quad F_\alpha = \nabla_{\boldsymbol{X}_\alpha} \chi_\alpha(\boldsymbol{X}_\alpha, t).$$

We introduce the following quantities for the constituent $\alpha \in \{1, \ldots, N\}$:

ρ_α	mass density of constituent α
T_α	stress tensor of constituent α
$\boldsymbol{b}_\alpha$	external body force on constituent α
ε_α	internal energy density of constituent α
$\boldsymbol{q}_\alpha$	energy flux of constituent α
r_α	external energy supply of constituent α
η_α	entropy density of constituent α
$\boldsymbol{\Phi}_\alpha$	entropy flux of constituent α
s_α	external entropy supply of constituent α
τ_α	mass production of constituent α
$\boldsymbol{m}_\alpha$	interaction force on constituent α
M_α	interaction moment of momentum on constituent α
l_α	energy production of constituent α
σ_α	entropy production of constituent α

2.1. Balance laws of each constituent

Following the pioneering work of Truesdell (1984), the basic laws of a mixture are given by the following balance equations for mass, linear momentum, moment of momentum, and energy for each constituent:

$$
\begin{aligned}
&\frac{\partial \rho_\alpha}{\partial t} + \mathrm{div}(\rho_\alpha \boldsymbol{v}_\alpha) = \tau_\alpha, \\
&\frac{\partial \rho_\alpha \boldsymbol{v}_\alpha}{\partial t} + \mathrm{div}(\rho_\alpha \boldsymbol{v}_\alpha \otimes \boldsymbol{v}_\alpha - T_\alpha) - \rho_\alpha \boldsymbol{b}_\alpha = \boldsymbol{m}_\alpha, \\
&T_\alpha - T_\alpha^T = M_\alpha, \\
&\frac{\partial}{\partial t}\left(\rho_\alpha \varepsilon_\alpha + \frac{1}{2}\rho_\alpha \boldsymbol{v}_\alpha^2\right) + \mathrm{div}\left(\left(\rho_\alpha \varepsilon_\alpha + \frac{1}{2}\rho_\alpha \boldsymbol{v}_\alpha^2\right)\boldsymbol{v}_\alpha + \boldsymbol{q}_\alpha - T_\alpha^T \boldsymbol{v}_\alpha\right) - \rho_\alpha r_\alpha - \rho_\alpha \boldsymbol{b}_\alpha \cdot \boldsymbol{v}_\alpha = l_\alpha,
\end{aligned}
$$
(2.1)

If the mixture reduces to a single constituent, the right hand side of the above equations are zero, and we recover the balance laws for a single body. The terms on the right hand side thus represent the physical transfers among different constituents, such as chemical reactions, interaction forces and energy transfer. In a mixture as a whole, we assume that such physical transfers are solely due to exchanges among constituents. Therefore, we postulate that the mixture as a whole should behave like a single body. This is expressed by the following relations:

$$\sum_\alpha \tau_\alpha = 0, \quad \sum_\alpha \boldsymbol{m}_\alpha = 0, \quad \sum_\alpha M_\alpha = 0, \quad \sum_\alpha l_\alpha = 0.$$
(2.2)