International Journal of Engineering Science 83 (2014) 162–173

Contents lists available at ScienceDirect

International Journal of Engineering Science

journal homepage: www.elsevier.com/locate/ijengsci

Variational method for optimal multimaterial composites and optimal design

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A R T I C L E I N F O

Article history:
Received 2 February 2014
Accepted 10 March 2014
Available online 15 May 2014

Keywords:
Structural optimization
Multimaterial composites
Optimal composites
Quasiconvex envelope
Multimaterial design
Nonconvex variational problems

A B S T R A C T

The paper outlines novel variational technique for finding microstructures of optimal multimaterial composites, bounds of composites properties, and multimaterial optimal designs. The translation method that is used for the exact two-material bounds is complemented by additional pointwise inequalities on stresses in materials within an optimal composites. The method leads to exact multimaterial bounds and provides a hint for optimal structures that may be multi-rank laminates or, for isotropic composites, "wheel assemblages". The Lagrangian of the formulated nonconvex multiwell variational problem is equal to the energy of the best adapted to the loading microstructures plus the cost of the used materials; the technique improves both the lower and upper bounds for the quasiconvex envelope of that Lagrangian. The problem of 2d elastic composites is described in some details; on particular, the isotropic component of the quasiconvex envelope of three-well Lagrangian for elastic energy is computed. The obtained results are applied for computing of optimal multimaterial elastic designs; an example of such a design is demonstrated. Finally, the optimal "wheel assemblages" are generalized and novel types of exotic microstructures with unusual properties are described.

1. Introduction

Modern technologies of microfabrication and 3d printing allow for a huge variety of structures to be manufactured for roughly the same price. Naturally, the material scientists want to know what is "the best" structure, or how composites microstructures can be optimized; these questions are also related to metamaterials that utilize various extreme properties. A close problem is the range of improvement of overall composite properties that can be achieved by varying the structure. There is no boundary between optimal design and an optimal composite material, which is also a structure at the microlevel: optimal designs are made from optimal composites. So far, the vast majority of related results deals with two-material composites because of theoretical limitations. Meanwhile, numerous applications call for optimal design of multimaterial composites, or even of porous composites made of two materials and void. Such designs are crucial for multi-physics applications, i.e. piezo-magnetic and electromagnetic devices, in metamaterials and adaptive structures.

Optimal microstructures of multimaterial composites differ drastically from two-material ones. The latter have a steady and intuitively expected topology: a strong material always surrounds weak inclusions, as in Hashin–Shtrikman coated

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http://dx.doi.org/10.1016/j.ijengsci.2014.03.002
0020-7225/

circles and second-rank laminates which may degenerate to simple laminates. In contrast, optimal three-material structures (Cherkaev, 2011; Cherkaev & Dzierżanowski, 2013; Cherkaev & Zhang, 2011) (Figs. 1 and 2) show a large variety of patterns and the optimal topology depends on the volume fractions. Optimal structures are diverse; they may or may not contain a strong envelope, and they may contain "hubs" of intermediate material connected by anisotropic "pathways" – laminates from the strong and weak materials, envelopes, and other configurations that reveal a geometrical essence of optimality. These structures are not unique, as shown in Figs. 1 and 2.

Obviously, the methods for finding them differ from the already developed methods used for optimal two-material structures. In this paper, we outline methods for determination of multimaterial optimal elastic composites and designs from them. The exposition is partially based on results obtained in Cherkaev (2009), Cherkaev (2011), Cherkaev and Dzierżanowski (2013), Cherkaev and Pruss (2012), Cherkaev and Zhang (2011) and Briggs, Cherkaev, and Dzierżanowski (in press).

2. Problems about optimal composites

2.1. The problem

The problem of the structure of optimal multimaterial composite has been studied for several decades. The bounds for multimaterial composites problem have been investigated starting from the papers by Hashin and Shtrikman (1963), Lurie and Cherkaev (1985), Milton (1981) and Milton and Kohn (1988). In Nesi (1995) suggested bounds for multimaterial mixtures that are better than Hashin–Shtrikman bounds; Gibiansky and Sigmund (2000) and Liu (2011) found new optimal multimaterial structures. In the past few years (2009–2012), we suggested (Cherkaev, 2009; Cherkaev, 2011; Cherkaev & Dzierżanowski, 2013; Cherkaev & Zhang, 2011) a new approach for optimal bounds of multimaterial mixtures and tested it on several examples of conducting composites.

Here we consider a problem about two-dimensional multiphase composites of a minimal compliance i.e. of a maximal stiffness. Assume that a unit periodic square cell $\Omega \subset R_2$ ($\|\Omega\| = 1$) is subdivided into N parts $\Omega_1, \ldots \Omega_N$ of given areas

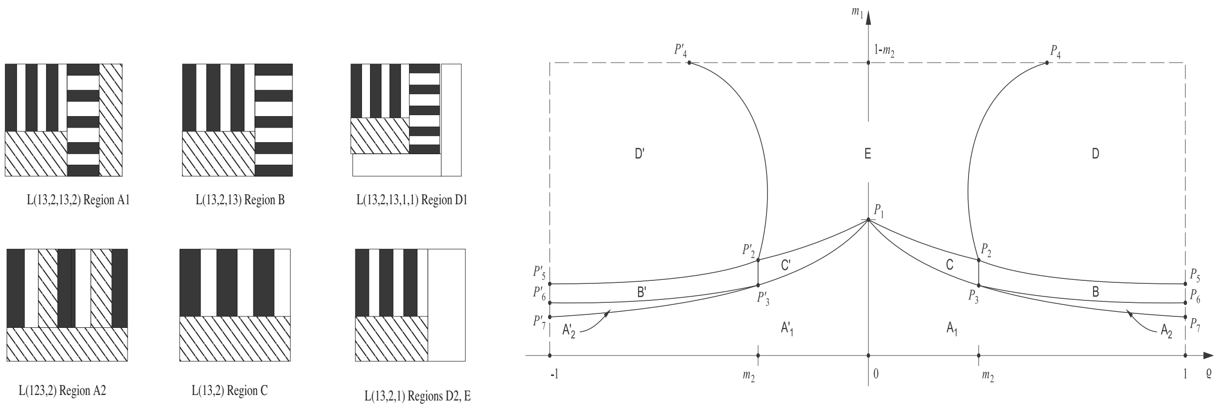

Fig. 1. Left: cartoon of optimal multi-rank laminates that minimize elastic energy (compliance) of a three-material composite, see Cherkaev and Dzierżanowski, 2013; Cherkaev and Zhang, 2011. The parameters and the types A–E of the structures depend on the volume fractions and the ratio p of eigenvalues of the applied external stress σ_0. Black fields denote void (an infinite compliance, $\kappa_3 = \infty$), striped fields denote a material of intermediate compliance κ_2 and white fields demote the stiffest material $\kappa_1, \kappa_1 < \kappa_2$. The notation L (13, 2, 13) shows the order of laminating as follows: materials κ_1 and κ_3 are laminated first, than they are laminated with material κ_2 in a orthogonal direction, then again laminated in an orthogonal direction with κ_1-κ_3 laminate. Right: regions of optimality of the structures A–E in dependence of the volume fraction m_1 of the best material (vertical axis) and p (horizontal axis) (Cherkaev & Dzierżanowski, 2013). Volume fraction of κ_2 is fixed. The right vertical line corresponds to uniform pressure, the center vertical line corresponds to uniaxial load, the left vertical line corresponds to pure shear load, see below, Section 4.

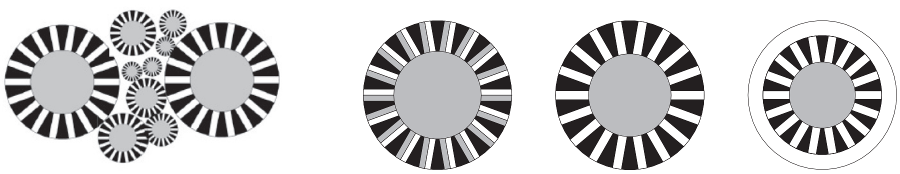

Fig. 2. An alternative optimal wheel-type assemblage for optimal isotropic microstructures (left) and elements W1, W2, W3 of the assemblage in dependence on volume fractions, see (Cherkaev, 2011). The black field here denotes void, the grey denotes material κ_2, and the white denotes material κ_1. The increase of the fraction m_1 of κ_1 (from left to right) leads to two topological transitions. The bulk modulus of the assemblages is equal to the bulk modulus of corresponding laminates in Fig. 1 made from the same materials taken in the same proportions.