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# Finite element method in equilibrium problems for a nonlinear shallow shell with an obstacle

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Dedicated to Prof. Leonid M. Zubov on the occasion of his 70th birthday.

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ABSTRACT

The purpose of this paper is to use a weak setup to justify application of the finite element method (FEM) to the equilibrium problem for a nonlinear model of a shallow shell clamped along part of an edge constrained by a frictionless obstacle. A suitable energy space is constructed and the generalized (weak) solutions are introduced. The obstacle condition is represented by a linearized model, and convergence of approximate FEM solutions to a weak solution is established. In particular, existence of a weak solution to the problem is proved. The result essentially extends that obtained in the paper Lebedev and Neymark (2006).



## 1. Introduction

Linear approximations in elasticity have been used for materials under small deformation. While this suffices for many purposes, non-small deformations require nonlinear approaches. The nonlinear mathematical theory of elasticity remains largely open, and the convergence of approximated solutions cannot be insured in general. However the nonlinear theory of shallow shells was thoroughly studied by Vorovich (1999), and convergence of the finite element method (FEM) is a known result (Vorovich & Lebedev, 1993). In some applications we must include additional constraints for displacements; this is the case for an obstacle whose contact area with the shell is unknown in advance. Owing to the practical importance of this problem, it has begun to receive intensive study (Lebedev & Neymark, 2006; Haslinger & Lovíšek, 1982; Bielski & Telega, 1998; Léger & Miara, 2011). The problem setup proceeds via Lagrange's principle of minimum total potential energy over the set of functions restricted by the condition of non-penetration of the obstacle.

FEM is a variational approach whose convergence is studied in this paper for the equilibrium problem describing a shallow shell with an obstacle. The Ritz method is used to introduce approximate solutions. Although convergence of approximate FEM solutions is well known for linear problems involving continuous and coercive bilinear forms Ciarlet (1978), the problem of convergence for nonlinear energy functionals is still an active field of research (Mansfield, 1981; Hosseini, Naghdabadi, & Jabbarzadeh, 2008; Kundu & Han, 2009; Cho & Rhee, 2012).

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The plan of the paper is as follows. First we present the relations and equations of nonlinear shallow shell theory, and discuss the mathematical model for an obstacle. Introducing the shell–obstacle equilibrium problem as the problem of minimizing the total potential energy of the system, we discuss the properties of the appropriate solution space. Finally, we present the principal ideas needed to solve the minimization problem using FEM, demonstrate the existence of the approximate FEM solutions, and establish convergence of these approximations to a weak solution of the problem.

## 2. Model of a shallow shell with an obstacle

### 2.1. Shallow shell model

A shell occupies a three-dimensional volume determined by a surface $S^*$ and its normals extending to length $h$ away from both sides of $S^*$. That is, the shell is the set of points $\boldsymbol{\varphi}(x) + \theta \boldsymbol{a}_3(x)$, where $\theta \in [-h, h]$ and $x = (\xi^1, \xi^2)$ are the intrinsic coordinates in $S^*$. We call $S^*$ the midsurface of the shell and $2h$ its thickness. Suppose $S^*$ is the image of an open set $\bar{\Omega} \subset \mathbb{R}^2$ under an injective mapping $\boldsymbol{\varphi} : \bar{\Omega} \to \mathbb{R}^3$ in $\boldsymbol{C}^2(\Omega)$. We also assume $\Omega$ is bounded and connected, with a piecewise differentiable boundary. Furthermore, the vectors $\partial_\alpha \boldsymbol{\varphi} = \boldsymbol{a}_\alpha$, where $\partial_\alpha$ denotes the partial derivative with respect to $\xi^\alpha$, are linearly independent, being a covariant basis of the tangent space at each point in $S^*$. The contravariant basis will be denoted by $\boldsymbol{a}^\alpha$. In what follows, Greek indices take values in the set $\{1, 2\}$ and Latin indices take values in $\{1, 2, 3\}$. We complement the covariant basis $\boldsymbol{a}_\alpha$ with the normal vector

$$\boldsymbol{a}^3 = \boldsymbol{a}_3 = \frac{\boldsymbol{a}_1 \times \boldsymbol{a}_2}{|\boldsymbol{a}_1 \times \boldsymbol{a}_2|}.$$

The components of the first and second fundamental forms of the midsurface are $a_{\alpha\beta} = \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta$ and $b_{\alpha\beta} = \boldsymbol{a}_3 \cdot \partial_\alpha \boldsymbol{a}_\beta$, respectively.

The area element of the midsurface is given by $d\Omega = \sqrt{a}\, d\xi^1\, d\xi^2$, where $a = \det(a_{\alpha\beta})$. We assume that $|h/R| \ll 1$, where $R$ is the minimum curvature of $S^*$, and that $a(x) \geqslant c > 0$ at every point $x \in \Omega$.

In what follows Einstein's summation rule over repeated sub-superscripts is used. The vector field $\boldsymbol{u} = u_\alpha \boldsymbol{a}^\alpha$ represents the displacement of each point in $S^*$ from its reference or initial configuration. It is known that under the Kirchhoff–Love hypothesis and shallowness of the shell, the rotation vector can be approximated as $\boldsymbol{\omega} = \partial_\alpha u_3 \boldsymbol{a}^\alpha$ so that the displacement of a point $\boldsymbol{\varphi}(x) + \theta \boldsymbol{a}_3(x)$ in the shell is given by

$$\boldsymbol{u}^\theta = \boldsymbol{u}(x) + \theta \boldsymbol{\omega}(x). \tag{1}$$

Hence it suffices to know the displacement of the midsurface, as displacements in the rest of the shell are already determined. The Christoffel symbols are $\Gamma^\gamma_{\alpha\beta} = \boldsymbol{a}^\gamma \cdot \partial_\beta \boldsymbol{a}_\alpha$, and the covariant derivatives of a vector field $\boldsymbol{u} = u_\alpha \boldsymbol{a}^\alpha$ in $S^*$ are defined by

$$u_{\alpha|\beta} = \partial_\beta u_\alpha - \Gamma^\gamma_{\alpha\beta} u_\gamma, \quad u_{3|\alpha} = \partial_\alpha u_3, \quad u_{3|\alpha\beta} = \partial_{\alpha\beta} u_3 - \Gamma^\gamma_{\alpha\beta} \partial_\gamma u_3.$$

The strain tensor of the midsurface $\gamma_{\alpha\beta}(\boldsymbol{u})$ and the tensor of change of curvature $\rho_{\alpha\beta}(\boldsymbol{u})$ for the shallow shell model are given by Vorovich (1999)

$$\gamma_{\alpha\beta}(\boldsymbol{u}) = \frac{1}{2}(u_{\alpha|\beta} + u_{\beta|\alpha}) - b_{\alpha\beta} u_3 + \frac{1}{2} \partial_\alpha u_3 \partial_\beta u_3, \tag{2}$$

$$\rho_{\alpha\beta}(\boldsymbol{u}) = u_{3|\alpha\beta}. \tag{3}$$

Assuming that Hooke's law holds, the tangential stress tensor $n^{\alpha\beta}(\boldsymbol{u})$ and the bending moments $m^{\alpha\beta}(\boldsymbol{u})$ are

$$n^{\alpha\beta}(\boldsymbol{u}) = h E^{\alpha\beta\lambda\mu} \gamma_{\lambda\mu}(\boldsymbol{u}),$$

$$m^{\alpha\beta}(\boldsymbol{u}) = \frac{1}{3} h^3 E^{\alpha\beta\lambda\mu} \rho_{\lambda\mu}(\boldsymbol{u}),$$

where $E^{\alpha\beta\lambda\mu} = E^{\lambda\mu\alpha\beta}$ is the tensor of elastic moduli for plane elasticity. For isotropic and homogeneous bodies, the tensor $E^{\alpha\beta\lambda\mu}$ depends on the geometry of the shell, Young's modulus $E$, and Poisson's ratio $\nu$ via the relation

$$E^{\alpha\beta\lambda\mu} = \frac{E}{1+\nu}\left(a^{\alpha\lambda} a^{\beta\mu} + a^{\alpha\mu} a^{\beta\lambda} + \frac{2\nu}{1-\nu} a^{\alpha\beta} a^{\lambda\mu}\right).$$

We assume in general that the functions $E^{\alpha\beta\lambda\mu}$ are bounded and piecewise continuous in $\Omega$ and that the tensor of elastic moduli is uniformly positive definite, i.e., there exists a constant $c > 0$ such that

$$E^{\alpha\beta\lambda\mu} \gamma_{\alpha\beta}(\boldsymbol{u}) \gamma_{\lambda\mu}(\boldsymbol{u}) \geqslant c \gamma_{\alpha\beta}(\boldsymbol{u}) \gamma^{\alpha\beta}(\boldsymbol{u})$$

for any symmetric tensor $\gamma_{\alpha\beta}$ at each point of $\Omega$.