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Stability of an inflated hyperelastic membrane tube with localized wall thinning

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Dedicated to Professor Leonid M. Zubov on the occasion of his 70th birthday.

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ABSTRACT

It is now well-known that when an infinitely long hyperelastic membrane tube free from any imperfections is inflated, a transcritical-type bifurcation may take place that corresponds to the sudden formation of a localized bulge. When the membrane tube is subjected to localized wall-thinning, the bifurcation curve would "unfold" into the turning-point type with the lower branch corresponding to uniform inflation in the absence of imperfections, and the upper branch to bifurcated states with larger amplitude. In this paper stability of bulged configurations corresponding to both branches is investigated with the use of the spectral method. It is shown that under pressure control and with respect to axi-symmetric perturbations, configurations corresponding to the lower branch are stable but those corresponding to the upper branch are unstable. Stability or instability is established by demonstrating the non-existence or existence of an unstable eigenvalue (an eigenvalue with a positive real part). This is achieved by constructing the Evans function that depends only on the spectral parameter. This function is analytic in the right half of the complex plane where its zeroes correspond to the unstable eigenvalues of the generalized spectral problem governing spectral instability. We show that due to the fact that the skew-symmetric operator $\mathcal{J}$ involved in the Hamiltonian formulation of the basic equations is onto, the zeroes of the Evans function can only be located on the real axis of the complex plane. We also comment on the connection between spectral (linear) stability and nonlinear (Lyapunov) stability.

1. Introduction

This study is part of our recent research effort that explores the postulate that initiation of at least some acute aneurysms in human arteries may be modeled as a bifurcation phenomenon (Fu, Rogerson, & Zhang, 2012). This postulate is motivated by two main results. Firstly, assuming that the artery is axisymmetric and homogeneous, and that the initial wall thickness is a constant, a localized bulge may form when the internal pressure reaches a certain critical value even if the pressure versus volume curve does not have a maximum in uniform inflation (Fu, Pearce, & Liu, 2008). Secondly, when imperfections such as localized wall weakening is introduced, the bifurcation pressure may fall to within the physiologically possible range

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(Fu & Xie, 2012). It is further postulated that it is after the initiation of such a localized bulge that biological processes such as remodelling take over, which in turn leads to further growth and final rupture of the aneurysm. The present study is also closely related to studies of solitary waves in hyperelastic membrane tubes; for a review of the relevant literature we refer to Fu and Il'ichev (2010). On the one hand, a static localized bulge can be viewed as a solitary wave that has zero propagation speed, the zero speed being induced by the internal pressure in the membrane tube. On the other hand, solitary waves, and more generally nonlinear waves, may play an important role in interrogating the health status of arteries (e.g. presence of an aneurysm) through signal processing (Noubissié, Kraenkel, & Woafo, 2009; Parker, 2009).

Identifying an aneurysm with a static localized bulge implies that this configuration must be stable since otherwise it cannot be observed. It was found that in the homogeneous case although internal fluid inertia would reduce the growth rate of the single unstable mode significantly, it alone cannot stabilize the unstable mode completely (Ilichev & Fu, 2012). Stabilization of the exponential growth of the aneurysm solution takes place in the presence of a non-zero mean flow (Fu & Il'ichev, in press), but due to the translational symmetry of the problem the standing bulging configuration is only orbitally stable or stable in form (Grillakis, Shatah, & Strauss, 1987); it can still propagate axially with a non-zero speed under perturbations.

When wall weakening is introduced, the problem in question is no longer invariant under translations, and in this case we may speak about the usual stability of the standing configuration. In this paper we investigate the stability of bulging configurations corresponding to both branches of the bifurcation curve. In the absence of any imperfections, the lower branch would correspond to uniform inflation whose stability/bifurcation has previously been studied by Shield (1972), Haughton and Ogden (1979), Chen (1997), and Zubov and Sheidakov (2008), and the upper branch would correspond to large amplitude bifurcated solutions whose stability properties have recently been studied by Fu and Xie (2010). Our stability analysis is based on the construction of the Evans function that depends only on the spectral parameter. The function is analytic in the right half of the complex plane and has there zeroes coinciding with unstable eigenvalues. We demonstrate that the zeroes of the Evans function can only be located on the real axis of the complex plane. Therefore, we need only to establish behavior of this function on the real axis which is technically possible, and based on this behavior we may draw conclusions not only about spectral instability of the bulging configurations under consideration, but also about its stability. In other words, absence of zeroes of the Evans function on the positive real axis implies linear stability of the aneurysm solution. Moreover, the correspondence of the spectrum of the related spectral problem in linear stability analysis to the one in Lyapunov (nonlinear) stability analysis is established. The spectrum η of the linearized problem is related to the spectrum $-\alpha$ of the Hessian of the energy via the relation $\alpha = \rho\eta^2$,where ρ is the density of the tube material. Therefore, with the Hessian being a self-adjoint operator, η can only have real or purely imaginary values, the latter corresponding to the continuous spectrum. Linear instability is governed by the presence of a discrete spectrum.

The rest of the paper is divided into four sections as follows. After presenting the Hamiltonian form of the governing equations we discuss in Section 3 the construction of fully nonlinear bulging (aneurysm) solutions. We present the bifurcation diagram, reflecting the appearance of standing bulging solutions, and also a set of three first-order differential equations to be solved numerically to obtain the fully nonlinear bulging solutions. This is then followed by Section 4 where we discuss properties of the related spectral problem in the linear stability analysis. We construct the Evans function for both branches of the bifurcation diagram and examine its behavior on the real axis of the right half of the complex plane. According to the existence or absence of its zeroes conclusions about linear instability or stability of the aneurysm solutions in question are made. The paper is concluded in Section 5 with a brief discussion of the connections between linear spectral stability and nonlinear Lyapunov stability, and relevance of our results to the mathematical modelling of aneurysm initiation in human arteries.

2. Formulation

We consider the inflation of a cylindrical membrane tube that is assumed to be incompressible, isotropic, and hyperelastic. In its undeformed configuration, the tube wall has thickness H that is not necessarily a constant, but the average of its outer and inner radii, hereafter referred to simply as the radius R, is a constant. The tube is assumed to be infinitely long, and end conditions are imposed at infinity. We use cylindrical polar coordinates, and undeformed and deformed configurations are described by coordinates (R, Θ, Z) and (r, θ, z), respectively.

We assume that the axisymmetry is maintained throughout the entire deformation, and so the deformation has the general form $r = r(Z,t), \theta = \Theta, z = z(Z,t)$. The principal directions of the deformation correspond to the lines of latitude, the meridian and the normal to the deformed surface, and the principal stretches are given by

$$\lambda_1 = \frac{r}{R}, \quad \lambda_2 = (r'^2 + z'^2)^{\frac{1}{2}}, \quad \lambda_3 = \frac{h}{H}, \tag{2.1}$$

where a prime represents differentiation with respect to Z, and h denotes the deformed thickness.

The principal Cauchy stresses $\sigma_1, \sigma_2, \sigma_3$ in the deformed configuration for an incompressible material are given by

$$\sigma_i = \lambda_i \widehat{W}_i - p, \quad i = 1,2,3 \quad (\text{no summation}), \tag{2.2}$$

where $\widehat{W} = \widehat{W}(\lambda_1, \lambda_2, \lambda_3)$ is the strain-energy function, $\widehat{W}_i = \partial\widehat{W}/\partial\lambda_i$, and p is the pressure associated with the constraint of incompressibility. Utilizing the incompressibility constraint $\lambda_1\lambda_2\lambda_3 = 1$ and the membrane assumption of no stress through the thickness direction (i.e. $\sigma_3 = 0$), we find