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On Grioli's minimum property and its relation to Cauchy's polar decomposition

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A B S T R A C T

In this paper we rediscover Grioli's important work on the optimality of the orthogonal factor in the polar decomposition in an euclidean distance framework. We also draw attention to recently obtained generalizations of this optimality property in a geodesic distance framework.

1. Introduction

Every invertible matrix $F \in \mathbb{R}^{n\times n}$ can be uniquely decomposed into a product of a unitary matrix $R \in O(n)$ and a positive definite matrix U:

$$F = RU.$$

The roots of this "polar decomposition theorem" lie in Cauchy's work on elasticity (Cauchy, 1841). Finger gave it as an algebraic statement and ideas for a proof (Finger, 1892, Eq (25)), the brothers E. and F. Cosserat proved it (Cosserat & Cosserat, 1896, Section 6). Matrix notation and extension to the complex case are due to Autonne (1902), cf. Ericksen (1960, Section 43) and Truesdell and Toupin (1960, Section 35–37). (The result also holds for complex matrices and for non-square matrices (then upon loosing the uniqueness property of R), see e.g. Higham (2008, ch. 8)).

The unitary polar factor R plays an important role in geometrically exact descriptions of solid materials. In this case $R^T F = U$ is called the right stretch tensor of the deformation gradient $F = \nabla\varphi$ and serves as a basic measure of the elastic deformation ([Bîrsan & Neff, 2013; Neff & Chełmiński, 2007; Neff & Forest, 2007; Neff, 2005, 2006). Indeed, it is known that the strain energy density for isotropic materials must depend only on the stretch U in order to be frame-indifferent. Similar reasonings on objectivity lead to the result that the strain energy for isotropic second gradient materials must depend on the stretch U and on its spatial gradient (see dell'Isola, Sciarra, & Vidoli, 2009; dell'Isola & Seppecher, 1995, 1997). Orthogonal tensors and material frame indifference are also used to introduce natural strain measures for Cosserat continua including microrotations (see e.g. Pietraszkiewicz & Eremeyev, 2009a, 2009b). The polar decomposition has also been employed to naturally introduce strain

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measures in the non-linear theory of shells (see e.g. Pietraszkiewicz, Szwabowicz, & Vallée, 2008). A new exact method for the polar decomposition has been introduced in Zubov and Rudev (1996) and some numerical issues can be found in Bouby, Fortuné, Pietraszkiewicz, and Vallée (2005). For additional applications and computational issues of the polar decomposition see e.g. Golub and Van Loan (1996, Ch. 12), Nakatsukasa and Higham (2013), Byers and Xu (2008), Nakatsukasa, Bai, and Gygi (2010) and Nakatsukasa and Higham (2012).

The unitary polar factor can be characterized by its best-approximation property. For given F, it is the unique unitary matrix realizing

$$\inf_{Q} \|F - Q\|^2 = \inf_{Q} \|Q^T F - I\|^2 = \|\sqrt{F^T F} - I\|^2 = \|U - I\|^2 \tag{1}$$

over all unitary matrices Q, where $\|\cdot\|$ is an arbitrary unitarily invariant norm (Fan & Hoffmann, 1955).

Optimality of the unitary polar factor is presently shown even for the expression $\|\mathrm{Log}(Q^T F)\|$ in Lankeit, Neff, and Nakatsukasa (2013), a distance measure arising from geometric considerations, connected with geodesic distances on matrix Lie groups (see Neff, Nakatsukasa, & Fischle, in press; Bîrsan, Neff, & Lankeit, 2013; Neff, Eidel, Osterbrink, & Martin, in press; Neff, Eidel, Osterbrink, & Martin, submitted for publication). Here, Log is the (possibly multi-valued) matrix logarithm, i.e. a solution of $\exp(X) = Q^T F$. In contrast to $\|F - Q\|$ (cf. Neff, Fischle, & Münch, 2008), in this logarithmic expression symmetric and skew-symmetric part of the matrix norm can be weighted differently and the optimality of the polar factor still holds:

$$\min_{Q \in \mathrm{SO}(n)} \left(\mu \|\mathrm{sym}\,\mathrm{Log}(Q^T F)\|^2 + \mu_c \|\mathrm{skew}\,\mathrm{Log}(Q^T F)\|^2 \right) = \mu \|\log \sqrt{F^T F}\|^2$$

for $\mu > 0, \mu_c \geqslant 0$, whereas the unitary polar factor fails to minimize the weighted expression

$$\mu \|\mathrm{sym}(Q^T F - I)\|^2 + \mu_c \|\mathrm{skew}(Q^T F - I)\|^2, \quad 0 < \mu_c < \mu$$

in the Frobenius norm.

In this short note we would like to trace back the development on the optimality of the polar factor to its presumable roots, the work (Grioli, 1940) of Grioli, who shows the minimization property (1) in the important special case of (some expression equivalent to) the Frobenius matrix norm and dimension 3.

This work seems to have gone nearly unnoticed (but Bouby et al., 2005; Martins & Podio-Guidugli, 1980; Podio-Guidugli & Martins, 1979; Truesdell & Toupin, 1960) and certainly the matrix-analysis community seems not to be aware of it. For example, Higham (2008) refers to the work Fan and Hoffmann (1955) for the optimality property (this is quite natural when being concerned with all unitarily invariant norms), who in turn seem to be nescient of Grioli's work.

We will juxtapose Grioli's original work Grioli (1940), carefully translated from the original Italian paper by us, to a version with current notation. It will become clear that Grioli is showing even more: He considers Euclidean movements, not only linear transformations. Therefore, in his framework, it is not possible to consider weighted expressions.

The present paper may also serve a pedagogical purpose: fundamental results are always older than it appears (see e.g. Russo, 2004).

2. Grioli's original work and comments

Grioli starts by putting himself in the framework of finitely deforming bodies:

Let C_* be the reference configuration of an arbitrary continuous material system $\mathcal{S}$; C and C' the current configurations of $\mathcal{S}$ as a consequence of two different regular displacements S and S', P_* the generic point of C_*; P the corresponding of P_*.

We consider a domain C_*, an arbitrary point $\vec{p}_* \in C_*$ and diffeomorphisms

$$S : C_* \to C, \quad S' : C_* \to C'$$

and denote $\vec{p} = S(\vec{p}_*)$, $\vec{p}' = S'(\vec{p}_*)$. We then restrict our investigation to a small ball $c_* = B_\rho(\vec{p}_*)$, where the affine approximation (by the first terms of the Taylor expansion)

$$S(\vec{p}_* + h) \approx \vec{p} + \nabla S(\vec{p}_*) \cdot h \tag{2}$$

is sufficiently good.

Let then c_* be a sphere with center P_* and radius ρ very small, which must be intended to be fixed independently of P_*. More precisely, we will consider ρ to be so small that (correspondingly to any P_*) the displacements S and S' in c_* can be identified with the corresponding homogeneous displacements tangent in P_*. If the displacements S and S' were homogeneous, no limitation would exist for ρ.
With reference to the arbitrary point P_* it is common to define "local distance" of the two displacements S and S' the integral:

$$\boldsymbol{d}_{P_*} = \int_{c_*} |\boldsymbol{Q}'\boldsymbol{Q}|^2 \, \boldsymbol{d}C_*,$$