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# On pull-in instabilities of microcantilevers

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ABSTRACT

In this paper the static deflection and pull-in instability of electrostatically actuated microcantilevers is investigated based on the strain gradient theory. The equation of motion and boundary conditions are derived using Hamilton's principle and solved numerically. It is shown that the strain gradient theory predicts size dependent normalized static deflection and pull-in voltage for the microbeam while according to the classical theory the normalized behavior of the microbeam is independent of its size. The results of strain gradient theory are compared with those of classical and modified couple stress theories and also experimental observations. According to this comparison, the classical theory underestimates the stiffness of the microbeam and there is a gap between the results predicted by the classical theory and those observed in experiment. It is demonstrated that this gap can be reduced when utilizing the strain gradient theory.



## 1. Introduction

Electrostatically actuated microbeams can be considered as an important part of some MicroElectroMechanical Systems (MEMS) such as micropumps, accelerometers, microswitches and microresonators. Pull-in instability is one of the most important phenomena which should be considered in design and analysis of electrostatically actuated microsystems. This phenomenon is observed in experiment by some researchers such as Nathanson, Newell, Wickstrom, and Davis (1967) and Taylor (1968).

Many researchers have studied the pull-in instability of electrostatically actuated microbeams. For example, Mojahedi, Moghimi Zand, and Ahmadian (2010), utilized the homotopy perturbation method to investigate the static pull-in of electrostatically actuated microbeams. Experimental investigation and finite element modeling of static behavior and static pull-in of electrostatically actuated microcantilevers and microbridges are presented by Ballestra, Brusa, De Pasquale, Munteanu, and Somà (2009). Chowdhury, Ahmadi, and Miller (2005) developed a closed form model for the static pull-in voltage of electrostatically actuated microcantilevers. Bochobza-Degani and Nemirovsky (2002) proposed a two-degree of freedom model to investigate the static pull-in of microcantilevers. Rong, Huang, Nie, and Li (2004) used an energy-based approach to derive an analytical solution for the pull-in voltage of multilayer clamped–clamped microbeams.

The structures used in MEMS have the dimensions in order of microns and sub-microns. Many experiments performed on the mechanical behavior of microstructures revealed that the normalized behavior of microstructures which the classical theory predicts to be independent of the size of the structure, is size-dependent. In these experiments it is also observed that

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the mechanical stiffness of the microstructures is significantly larger than the stiffness value predicted by the classical theory. Some of these experimental researches can be mentioned as

- Torsion of thin copper wires by Fleck, Muller, Ashby, and Hutchinson (1994).
- Bending of nickel microbeams by Stolken and Evans (1998).
- Bending of microcantilevers made of epoxy by Lam, Yang, Chong, Wang, and Tong (2003).
- Deflection of microcantilevers made of polypropylene by McFarland and Colton (2005).

According to the abovementioned experimental researches, the classical theory can neither model the size-dependent behavior of microstructures nor predict the accurate mechanical response of such components. To model and explain the size-dependent behavior of structures, non-classical continuum theories such as couple stress theory and strain gradient theory have been introduced.

The couple stress theory is developed by some researches such as Mindlin and Tiersten (1962), Toupin (1962) and Koiter (1964). This theory utilizes some new material parameters beside the classical material constants (i.e. elastic modulus and Poisson's ratio) to predict and model the size effects in microstructures. This theory is successfully utilized to model the size-dependent behavior of Timoshenko microbeams (Asghari, Kahrobaiyan, Rahaeifard, & Ahmadian, 2011). Yang, Chong, Lam, and Tong (2002) introduced a modified couple stress theory by employing the equilibrium equation of moment of couples beside the classical equilibrium equations (i.e. equilibrium equation of forces and moment of forces). The modified couple stress theory has been utilized to develop the size-dependent formulation of microbeams and microplates (Asghari, Kahrobaiyan, & Ahmadian, 2010; Farokhi, Ghayesh, & Amabili, 2013; Şimşek & Reddy, 2013; Thai & Kim, 2013). Also based on this theory, the size-dependent behavior of microsystems has been investigated by some researchers (Akgöz & Civalek, 2011; Kahrobaiyan, Rahaeifard, & Ahmadian, 2014; Tang, Ni, Wang, Luo, & Wang, 2014; Wang, Xu, & Ni, 2013).

The strain gradient theory is introduced by Mindlin (1965). He considered the density of strain energy as a function of the strain tensor and its first and second derivatives. The formulation of the strain gradient theory is simplified by Fleck and Hutchinson (1993, 1997, 2001). They considered only the first derivative of the strain effective in the strain energy density and called this new theory the strain gradient theory. This theory could be reduced to the couple stress theory in special cases. Lam et al. (2003), modified and simplified the strain gradient theory in a similar way considered by Yang et al. (2002) and introduced the modified strain gradient theory. Henceforward in this paper wherever the strain gradient theory is mentioned it refers to the modified strain gradient theory developed by Lam et al. (2003). This new established theory is reduced to the modified couple stress theory in special cases. Many researches utilized the strain gradient theory to investigate the mechanical behavior of micro scale structures. Some of these works can be outline as follows.

- Static and dynamic behavior of Euler–Bernoulli microbeams by Kong, Zhou, Nie, and Wang (2009).
- Static and dynamic behavior of linear and nonlinear functionally graded microbeams by Kahrobaiyan, Rahaeifard, Tajalli, and Ahmadian (2012) and Rahaeifard, Kahrobaiyan, Ahmadian, and Firoozbakhsh (2013).
- Mechanical behavior of functionally graded sinusoidal microbeams Lei, He, Zhang, Gan, and Zeng (2013).
- Size-dependent yielding of micro scale structures by Rahaeifard, Ahmadian, and Firoozbakhsh (2014).

In this paper the static deflection and pull-in voltage of electrostatically actuated microcantilevers is investigated based on the strain gradient theory. The beam is modeled using Euler–Bernoulli beam theory and Hamilton's principle is utilized to derive the equation of static deflection and boundary conditions. Results of the strain gradient theory are compared with those of modified couple stress theory, classical theory and experimental observations. According to the results, when the ratio of the beam thickness to the material internal length scales is not large, there is a notable gap between the results given by classical theory and those of non-classical theories. It is also concluded that the strain gradient theory can reduce the gap between the theoretical results and experimental findings.

## 2. Preliminaries

According to the strain gradient theory the strain energy of a linear elastic material occupying region $\Omega$ can be written as (Lam et al., 2003)

$$U_m = \frac{1}{2}\int_{\Omega}\left(\sigma_{ij}\varepsilon_{ij} + p_i\gamma_i + \tau^{(1)}_{ijk}\eta^{(1)}_{ijk} + m^s_{ij}\chi^s_{ij}\right)dv \tag{1}$$

in which

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{2}$$

$$\gamma_i = \varepsilon_{mm,i} \tag{3}$$

$$\eta^{(1)}_{ijk} = \frac{1}{3}(\varepsilon_{jk,i} + \varepsilon_{ki,j} + \varepsilon_{ij,k}) - \frac{1}{15}\delta_{ij}(\varepsilon_{mm,k} + 2\varepsilon_{mk,m}) - \frac{1}{15}\left[\delta_{jk}(\varepsilon_{mm,i} + 2\varepsilon_{mi,m}) + \delta_{ki}(\varepsilon_{mm,j} + 2\varepsilon_{mj,m})\right] \tag{4}$$