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Mathematical models for fluids with pressure-dependent viscosity flowing in porous media

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ABSTRACT

In this paper we study three filtration problems through porous media, assuming that the viscosity of the fluid depends on pressure. After showing that in this case Darcy's law is "formally" preserved (meaning that the formal relation remains unchanged except for viscosity that now depends on pressure), we focus on the following problems: Green–Ampt infiltration through a dry porous medium; the Dam problem; the Muskat problem. For each model (free boundary problems) we obtain explicit solutions that allow to quantify the detachment from the classical case, where with the word "classical" we mean that viscosity is taken constant.

1. Introduction

Fluids with viscosity depending on pressure have recently drawn a lot of attention from the scientific community. In the last decades a remarkable number of experimental papers has been produced to support the claim that viscosity may vary with pressure (even if the fluid remains incompressible), proving that in many cases it is imperative to take such a dependence into account (Binding, Chouch, & Walters, 1998; Goubert, Vermant, Moldenaers, Gotffert, & Ernst, 2001; Johnson & Tevaarwerk, 1977). Of course many empirical models have been proposed (Denn, 1981), many of which take also into account the dependence on temperature.

Concerning the empirical formula adopted (see, e.g., Szeri (1998)) we indicate the linear law

$$\mu(P) = \alpha P, \quad \text{or} \quad \mu(P) = \mu_o(1 + \alpha P) \tag{1}$$

and the exponential (or Barus) law

$$\mu = \mu_o e^{\alpha P}. \tag{2}$$

Some other possibilities can be found, for instance, in Malek, Necas̆, and Rajagopal (2002, 2002).

The flow of a fluid through a porous medium is typically described by means of the well known Darcy's law, which gives a linear relation between the discharge and the pressure gradient. Darcy's law can be derived through an homogenization procedure considering the Stokes flow at the micro-scale and then upscaling the system to the macro-scale (see, for instance, Mikelić (2000) and Chamsri (2013)). This derivation is well known when one deals with a viscous incompressible/compressible fluid, and it has been recently studied for the case of a fluid with pressure dependent viscosity (Savatorova & Rajagopal,

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2011). We also refer the reader to Srinivasan and Rajagopal (2014) for a general thermodynamic framework (based on the criterion of maximal rate of entropy production) that can be used to derive Darcy and Brinkman models and their generalizations.

In this paper we consider Darcy's law with pressure-dependent viscosity and apply such modified version to some classical filtration problems. In particular we show that "formally" Darcy's law remains the same, namely

$$\mathbf{q} = -\frac{\mathbf{k}}{\mu(P)}(\nabla P - \mathbf{f}) \tag{3}$$

and use (3) to investigate three classical models: Green Ampt model (Section 2); the Dam problem (Section 3); the Muskat problem (Section 4). We show that explicit solutions can be found and illustrate the qualitative behavior of such solutions. The use of (3) allows indeed to extend models that have been investigated using the classical Darcy's law to the case of pressure dependent viscosity. Interesting applications have been described in Srinivasan, Bonito, and Rajagopal (2013) and in Nakshatrala and Rajagopal (2011). Special flows of fluid with pressure depend viscosity (even not strictly related to filtration) have been recently studied in detail (Fusi, Farina, & Rosso, 2014; Rajagopal, Saccomandi, & Vergori, 2012).

Eq. (3) was first obtained via homogenization in Savatorova and Rajagopal (2011). The procedure consists in determining the flow in a periodic cell $\Omega = \Omega_s \cup \Omega_\ell$ formed by a solid and a liquid part, assuming that the stress tensor in the fluid phase is given by $\mathbf{T} = -P\mathbf{I} + 2\mu(P)\mathbf{D}(\mathbf{u})$, where $\mathbf{u}$ is the velocity field,

$$\mathbf{D}(\mathbf{u}) = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right), \tag{4}$$

is the symmetric part of the velocity gradient $\nabla\mathbf{u}$, P is pressure and μ(viscosity) is a smooth bounded function of P. Assuming incompressibility and creeping flow with no-slip on the solid boundary Γ_s we write the governing equations as

$$\begin{cases} \nabla\cdot\mathbf{u} = 0, & \text{in } \Omega_\ell, \\ -\nabla P + \mu\Delta\mathbf{u} + 2\mu'\mathbf{D}[\nabla P] = -\mathbf{f}, & \text{in } \Omega_\ell, \\ \mathbf{u} = 0, & \text{on } \Gamma_s, \end{cases} \tag{5}$$

where $\mathbf{f}$ represents the body force vector. The macroscopic variable of problem (5) is $\mathbf{x}$, while the microscopic variable is given by $\varepsilon\mathbf{y} = \mathbf{x}$. Writing the classical expansion (Bakhvalov & Panasenko, 1989)

$$\mathbf{u} = \varepsilon^\alpha \sum_{k=0}^{\infty} \varepsilon^k \mathbf{u}^{(k)}(\mathbf{x},\mathbf{y}), \quad P = \varepsilon^\beta \sum_{k=0}^{\infty} \varepsilon^k P^{(k)}(\mathbf{x},\mathbf{y}), \tag{6}$$

where $\mathbf{u}^{(k)}$ and $P^{(k)}$ are Ω-periodic and α, β are parameters yielding physically meaningful solutions, problem (5) can be reformulated (at the leading order) as

$$\begin{cases} \nabla_y\cdot\mathbf{u}^{(0)} = 0, & \text{in } \Omega_\ell, \\ -\nabla_x P^{(0)} - \nabla_y P^{(1)} + \mu^{(0)}\Delta_y\mathbf{u}^{(0)} + \mathbf{f} = 0, & \text{in } \Omega_\ell, \end{cases} \tag{7}$$

where $\mu^{(0)}$ is $\mu(P^{(0)})$ and where $P^{(0)} = P^{(0)}(\mathbf{x})$ does not depend on the microscopic coordinates. Problem (7) is then rewritten in a weak form whose existence and uniqueness is guaranteed by the Lax–Milgram theorem, yielding

$$\mathbf{u}^{(0)} = \frac{1}{\mu^{(0)}}\sum_{i=1}^{3}\left(f_i - \frac{\partial P^{(0)}}{\partial x_i}\right)\mathbf{u}_i, \tag{8}$$

where $\mathbf{u}_i$ is the solution of the auxiliary problem

$$\begin{cases} \nabla_y\cdot\mathbf{u}_i = 0, & \text{in } \Omega_\ell, \\ -\nabla_y m_i + \Delta_y\mathbf{u}_i + \mathbf{e}_i = 0, & \text{in } \Omega_\ell, \quad i = 1,2,3, \\ \mathbf{u}_i = 0 & \text{on } \Gamma_s, \end{cases} \tag{9}$$

with m_i is Ω-periodic and $\mathbf{e}_i$ is the unit vector along y_i. After extending $\mathbf{u}^{(0)}$ and $\mathbf{u}_i$ to zero in the solid phase of Ω and defining

$$\mathbf{q} = \frac{1}{|\Omega|}\int_{\Omega_\ell}\mathbf{u}^{(0)}d\mathbf{y}, \quad \mathbf{k}_i = \frac{1}{|\Omega|}\int_{\Omega_\ell}\mathbf{k}_i d\mathbf{y}. \tag{10}$$

we get the "modified" Darcy's law (3), with $\mathbf{k}$ denoting the tensor with entries $(\boldsymbol{\kappa})_{i,j} = \mathbf{k}_i\cdot\mathbf{e}_j$.

2. Application to the Green–Ampt infiltration model

As a first example we study how the use of relation (3) modifies the classical Green–Ampt infiltration model (Mikelić, 2000). For the sake of simplicity we limit ourselves to consider the one dimensional setting depicted in Fig. 1, assuming that a dry medium occupying the region $z < 0$ is penetrated by a fluid supplied at the surface $z = 0$. We assume that the motion is