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Three-phase model for a composite material with cylindrical circular inclusions. Part I: Application of the boundary shape perturbation method

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ABSTRACT

The problem of determining the effective thermal conductivity of a composite material with periodic cylindrical inclusions of a circular cross-section arranged in a square grid is analyzed. Defining mathematical relationships are derived on the basis of a three-phase composite model, asymptotic homogenization technique and application of the boundary shape perturbation method to solve the derived unit cell problems. The analytical expression for the effective coefficient of thermal conductivity is obtained in the zero-order approximation and the correction to this expression is derived in the first-order approximation. This correction allows taking into account the geometry of inclusion, not just its volume fraction. It is shown that a small ε_1-order perturbation of the unit cell contour yields the ε_1^2-order contribution to the homogenized relations. The obtained solution is analyzed and compared with known results in some particular cases, and the limits of its applicability are evaluated.

1. Introduction

The classic three-phase model is described in detail and applied to solve various problems in the mechanics of composites in a number of reviews and monographs. At first time a three-phase model (TPhM) for a composite material was used by Kerner (1956) and van der Poel (1958). The essence of this model is that entire periodic composite structure, with the exception of one cell, is replaced by a homogeneous medium with some unknown effective characteristics. Further the unknown effective parameters are determined from the relations derived from the energy principle stating that the energies stored in the composite material and the equivalent homogeneous medium are equal. Subsequently TPhM was applied to solve problems for composites with cylindrical and spherical inclusions, see, e.g., Christensen (2005) and Christensen and Lo (1979).

For an elastic composite medium with elastic spherical inclusions of variable sizes, Hashin (1962) proposed a model in which it was assumed that sizes of inclusions are not random but they obey a rule that the ratio of the radius of inclusion to the radius of surrounding matrix is uniform for each inclusion regardless of its absolute size. That means that the distribution of sizes of inclusions should be such that the whole volume of composite material is filled with inclusions with the

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uniform volume fraction, which implies that the sizes of inclusions decrease to infinitesimal. Three-phase composite model is further applied to find an effective shear modulus of such a composite material, see Christensen (2005).

For a fibre-reinforced composite with random dimensions of cross sections of cylindrical inclusions, Hashin and Rosen (1964) introduced so-called polydisperse composite model. This model is a two-dimensional analog of three-dimensional model of a polydisperse composite medium with spherical inclusions. However, the polydisperse model does not allow determining all the effective characteristics of transversely isotropic composite medium. Therefore, TPhM was applied to obtain an exact solution for the effective shear modulus in the plane of isotropy, see Christensen (2005).

The classical TPhM is also described in Milton (2002), Torquato (2002) and Buryachenko (2001).

Note also the conceptually close effective medium approximation (EMA) introduced by Bruggeman (1935). The effective medium approximation is a classical micromechanical model used in derivation of the effective properties of composite materials. It is shown that the EMA is realizable, and therefore it always satisfy the Hashin–Shtrikman bounds, see Benveniste and Milton (2010a, 2010b).

Asymptotic justification of the TPhM is presented in Andrianov, Danishevs'kyy, and Kalamkarov (2006). Different aspects of combination of the TPhM with the asymptotic homogenization method were developed and applied to the analysis of periodic inhomogeneous materials in Andrianov, Kalamkarov, and Starushenko (2013a, 2013b), Andrianov and Starushenko (1995a), Starushenko and Rogoza (2008).

TPhM and Padé approximants were used in Andrianov and Starushenko (1995b) and Andrianov, Starushenko, and Tokarzewski (1998, 1999) to determine the effective coefficient of thermal conductivity of two-phase composite structure with periodic cylindrical inclusions of a square cross-section arranged in a square grid. In this case, the idealization of the TPhM from a mathematical point of view leads to the problem of coupling matrix with an equivalent homogeneous medium in which the temperature and heat flux decay far away from the inclusion.

The present paper is dealing with the calculation of effective thermal conductivity of the composite material with periodic cylindrical inclusions of a circular cross-section arranged in a square grid. The following methodology is applied:

- Two-scale asymptotic homogenization method, see Bensoussan, Lions, and Papanicolaou (1978), Lions (1982), Bakhvalov and Panasenko (1989), Kalamkarov (1992, 2014), Kalamkarov and Kolpakov (1997) and Kalamkarov, Andrianov, and Danishevs'kyy (2009).
- The obtained unit cell problems are solved using Three-phase composite model, with the application of the
- Boundary shape perturbation method, see Guz and Nemish (1987) and Nemish (1989).

1.1. Application of the asymptotic homogenization method

Consider a two-phase inhomogeneous composite material consisting of a continuous matrix with the periodically distributed cylindrical inclusions of a circular cross-section. Assume that the material is doubly-periodic with the same period in both directions, and that the inclusions are located in a square grid, see Fig. 1.

The analysis of thermal conductivity of such composite structure is based on the Poisson equation with the corresponding boundary conditions and interphase relations. It should be noted that this mathematical model can have a number of different physical interpretations, see, e.g., Batchelor (1974). Therefore, the mathematical relationships and discussion related to

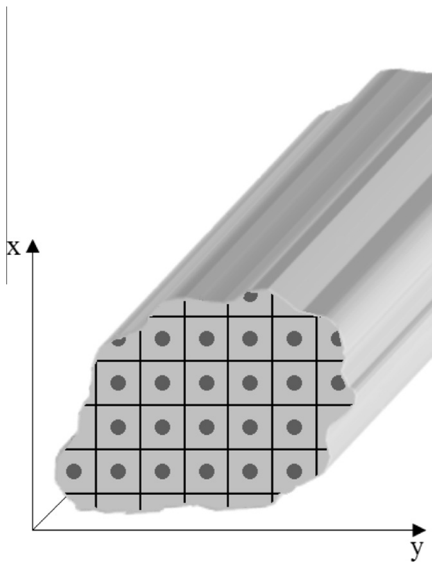

Fig. 1. Cross-section of a composite material with cylindrical inclusions of a circular cross-section.