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Continuum Mechanics and Lagrange equations with generalised coordinates

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ABSTRACT

The main aim of the actual problem is to obtain Lagrange equations when the chosen parameters do not respect material rigidity, so inducing strains (and Continuum Mechanics). The proposed method consist of two principal parts: first the definition of a family of generalised displacements involving strains and second the elimination of the Cauchy stress tensor in the Virtual Work Principle valuable in Continuum Mechanics. As a final statement the rigidity law is introduced on the parameters to complete the obtained equations. On a friction problem, it is highlighted the necessity to really distinguish between these mathematical compatibility conditions taking account of the nature of the material and other relations expressing some experimental boundary conditions like friction laws.

1. Introduction

In a series of several papers (Schutte & Udwadia, 2011; Udwadia & Kalaba, 2001; Udwadia & Schutte 2010), Udwadia et al. have published interesting papers concerning Multibody Dynamical Systems, with particular attention to the numerical treatment of mechanical equations. In 2011 they resumed their previous researches and proposed a study of a spacecraft system with numerical results. The main idea of their works concerns the treatment of the mechanical equations completed by constraint equations whatever the number of parameters, even if this last number is larger than the minimum required as it is the case with quaternions. Other recent results concerning Continuum Mechanics and Analytical Dynamics may be found in the two papers (O'reilly and Srinivasa, 2001; Rajagopal and Srinivasa, 2005) that propose other point of views. In the referenced papers, interesting introductions propose a review of earlier works on the Analytical Dynamics from the well-known Lagrange equations (Goldstein, 1950, Lagrange, 1787), but, since this subject is classical and known, it will not be repeated here.

However we will note some limitations of their results. First, they use equality constraints of the type $\varphi_i(q,\dot{q},t)=0$, $i=1,2,\ldots,m$, as in formula (17) of their paper (Schutte & Udwadia, 2011), where the above functions are smooth enough for twice time derivations. But in practical problems, friction leads to non smooth equations, in fact to variational inequalities associate to inequality constraint, except in the very particular case of no slipping (and no-friction): it is evident that if the constraints are unilateral, a theory that uses equality constraints cannot be apply. Second, even if these constraints are smooth enough, by example if a rigid body is divided, it is not clear why the virtual n-displacements w imply that the virtual work done by forces of constraints is written as $w^T\cdot C(q,\dot{q},t)$, where C is an n-vector describing the nature of the constraints.

A third (and essential) remark concerns the use of parameters. When rigidity is a priori included in the definition of parameters, as it is the case with Euler angles for rotational motions, then D'Alembert Principle gives the equations of

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motions by an implicit elimination of internal forces (i.e. stresses). But if rigidity is imposed by explicit relations, as it is the case with quaternions for rotational motions leading to one smooth constraint, why it is assumed (Udwadia & Schutte, 2010) that the virtual work associated to this rigidity constraint appears on the form $w^T \cdot C(q, \dot{q}, t)$? This hypothesis appears as undue because made without justification. Taking the example of the (q_0,q_3)-quaternions describing a two-dimensional rotation

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1-2q_3^2 & -2q_0q_3 & 0 \\ 2q_0q_3 & 1-2q_3^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}, \quad C = \left[1 + 4q_3^2(q_0^2 + q_3^2 - 1)\right] I_d \tag{0}$$

(x actual position; X initial position), why the relation $C = Id$ (or $q^Tq = 1$) requires that associate virtual work appears like $w^T \cdot C(q, \dot{q}, t)$ in order to take account of interior forces? In reality, as an example, the circle $X^2 + Y^2 = 1$ become the circle $x^2 + y^2 = 1 + 4q_3^2(q_0^2 + q_3^2 - 1)$ and this last evolution must be considered as a motion of some deformable (e.g. elastic) body if the above constraint $q^Tq = 1$ is not fulfilled. Then, if the above rigidity constraint is not taken account, it is clear that such parameters require Continuum Mechanics Principles, since strains (so Cauchy stresses) must be introduced. We conclude that D'Alembert Principles that govern rigid bodies motions only cannot be convenient for a priori deformable bodies.

These questions are fundamental. So, the present work proposes to make clear the origin of constraints, particularly if they express rigidity as in the case of quaternions or concern friction as in usual technological systems: clearly, the first one is a material constitutive law whereas the second one is a boundary condition. In the following, we highlight the origin of the mechanical equations, particularly the necessity to use the Virtual Work Principle (VWP) for a continuum with Cauchy stress tensor if rigidity is not implicitly satisfied (Souchet, 2004). Finally, as an example, we propose to illustrate the obtained results by obtaining equations in a problem involving contact with friction. In the following we propose three steps in our basic statement.

2. First step: Virtual Work Principle in Continuum Mechanics

It is natural to consider a single rigid body as a continuum B whose elements, called material points, constitute a three-dimensional manifold. A global configuration B_t (or B for simplicity of notations) of the body at time t is specified in an inertial coordinate frame $T_o = O_o\, x_oy_oz_o$ by some smooth function $x = \chi(X, t)$, where x is the actual position of the particle located at X at initial time; B_o is the initial configuration or some reference configuration. We note by O the point defined by $x = \chi(O_0, t)$.

In Classical Continuum Mechanics, two geometric definitions are introduced: first the deformation gradient $F = \partial\chi/dX$ that is a smooth, invertible and linear mapping, second the Cauchy-Green tensor $C = F^TF$ that is a measure of strain in the continuous body. Naturally F and C are tensors that represent local properties around the point X at time t. We note that the condition $J = detF > 0$ since at initial time we have $J_o = 1$. Kinematical quantities are also introduced, viz velocity $\dot{x} = \partial\chi/\partial t$ and acceleration $\ddot{x} = \partial^2\chi/\partial t^2$ on the initial configuration B_0, respectively $u(x,t)$ and $a(x,t)$ on the actual configuration B, satisfying the identity $gradu = (\partial F/\partial t)F^{-1}$ (note that time derivative is denoted by a point).

Contact forces developed in the interior of the body B are taken into account by the Cauchy stress symmetric tensor σ defined on the actual configuration B. The Virtual Work Principle (in brief VWP) relies acceleration and interior forces due to Cauchy stressses to exterior forces f on the volume B and φ on the surface Γ surrounding B, according satisfaction to the linear form

$$-\int_B \rho a.v dx + \int_B \rho f.v dx + \int_\Gamma \varphi.v da - \int_B \sigma : grad\, v dx = 0 \tag{1}$$

defined on the space V of (virtual) piecewise continuous "velocities" (or "displacements") v. In the above formula, ρ designs density, the point (.) is the scalar product of vectors and the double point (:) the scalar product of second order tensors. This formula is the basic statement of Classical Continuum Mechanics (Souchet, 2004), before introduction of constitutive laws and definition of boundary and initial value problems. In a first time we apply this principle to bodies whose deformation depends on independent parameters q_i functions of time t, i.e.

$$x = G(t, q(t); X), \quad q = (q_1, q_2, \dots q_n)^T \tag{2}$$

Finally we recall that this above principle VWP works for any part of the body and also for any system $B = (B_1, B_2, \dots, B_m)$ of bodies; as an example a body B may be divided into two sub-bodies.

3. Second step: generalised displacements

Now in Classical Dynamics, rigidity is a constitutive law of materials, so that the motion of such a single rigid body B is defined by some relation $x = T_0(t)+R(t)X$ where $T_0(t)$ is a translation and $R(t)$ a rotation in the three dimensional space; it is well known that T_0 depends generally on three coordinates, R depends also on three coordinates and satisfies $R^{-1} = R^T$ We note by O the point defined by $x = T_0(t)$, translated of the origin $X = 0$: on a practical manner we first define the point O, and then the rotation $x - T_0(t) = R(t)X$.