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A one-dimensional model for unsteady axisymmetric swirling motion of a viscous fluid in a variable radius straight circular tube

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ABSTRACT

A one-dimensional model for the flow of a viscous fluid with axisymmetric swirling motion is derived in the particular case of a straight tube of variable circular cross-section. The model is obtained by integrating the Navier–Stokes equations over cross section the tube, taking a velocity field approximation provided by the Cosserat theory. This procedure yields a one-dimensional system, depending only on time and a single spatial variable. The velocity field approximation satisfies exactly both the incompressibility condition and the kinematic boundary condition. From this reduced system, we derive unsteady equations for the wall shear stress and mean pressure gradient depending on the volume flow rate, the Womersley number, the Rossby number and the swirling scalar function over a finite section of the tube geometry. Moreover, we obtain the corresponding partial differential equation for the scalar swirling function.

1. Introduction

In this paper we present a one-dimensional model for the swirling motion of a viscous fluid, based on the Cosserat theory – also called director theory. The swirling features in flow fields are commonly called vortices. For most purposes (see e.g., Lugt, 1972; Kitoh, 2006; Nissan and Bressan, 1961; Moene, 2003), a vortex is characterized by a swirling motion of fluid around a central region. The swirling flow through a straight tube of variable circular cross-section is a complex turbulent flow and it is still challenging to predict and it is computationally demanding to simulate the full three-dimensional equations for swirling flows, which makes the direct 3D numerical simulation infeasible in many relevant situations. In recent years, the computational dynamics of two-dimensional swirling flows has been studied extensively with the purpose of better understanding the underlying physical phenomena and getting insight on important applications like the study of hurricanes and tornadoes (see e.g., Guinn and Shubert, 1993; Lewellen, 1993). Here we apply the Cosserat theory (see Caulk and Naghdi (1987)) to reduce the full three-dimensional system of fluid equations to a one-dimensional system of partial differential equations, which depend only on time and on a single spatial variable. The basis of this theory (see Duhem (1893) and Cosserat and Cosserat (1908)) is to consider an additional structure of deformable vectors (called directors) assigned to each point on a spatial curve (the Cosserat curve). The use of directors in continuum mechanics goes back to Duhem (1893), who regarded a body as a collection of points, together with associated directions. Theories based on such models of an oriented medium were further developed by Cosserat and Cosserat (1908). This theory has also been used by several authors in

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studies of rods, plates and shells (see e.g., Ericksen and Truesdell (1958), Truesdell and Toupin(1960), Green et al. (1968); Green et al., 1974 and Naghdi (1972)). An analogous hierarchical theory for unsteady/steady flows has been developed by Caulk and Naghdi (1987) in straight tubes of variable circular cross-section and by Green and Naghdi (1984) in channels. Applications to unsteady viscous flows in curved tubes of elliptic cross-section were presented by Green et al. (1993). Recently, this theory has been applied to several models arising in haemodynamics. We refer to a survey by Robertson and Sequeira (2005), and an application by Carapau and Sequeira (2006a). Also, Carapau and co-authors (see Carapau and Sequeira, 2006b, 2006c; Carapau et al., 2007; Carapau and Sequeira, 2008; Carapau, 2008a, 2008b; Carapau, 2009; Carapau, 2010a, 2010b) have analysed extensions of the theory to deal with several non-Newtonian fluid models in different geometries. Regarding the swirling motion, this hierarchical theory was used to study a Rivlin–Ericksen fluid (complexity $n = 2$) with axisymmetric swirling steady motion flowing in a straight tube of variable circular cross-section (see Carapau, 2009). This theory was validated in straight tubes of constant circular cross-section for Newtonian fluids (see Caulk and Naghdi, 1987) and for some non-Newtonian fluids (see Carapau and Sequeira, 2006a, 2006b). Another validation was provided in the case of a particular non-Newtonian fluid flow in a linearly tapered tube for (see Carapau, 2010a). The advantage of using a theory of directed curves is not so much getting an approximation of the three-dimensional system, but rather in using it as an independent framework to predict some properties of the full three-dimensional problem. The main features of the director theory are: (i) it incorporates all components of the linear momentum equations; (ii) it is a hierarchical theory, making it possible to increase the accuracy of the model; (iii) there is no need for closure approximations, i.e. additional relations between variables in the 1D model; (iv) invariance under superposed rigid body motions is satisfied at each order; (v) the wall shear stress enters directly in the formulation as a dependent variable and (vi) the director theory has been shown to be useful for modeling flow in curved tubes. A detailed discussion about the Cosserat theory can be found in Green and Naghdi (1993) and Green et al. (1993).

Using this director theory, we can intend to predict the main properties of a three-dimensional given problem, where the fluid three-dimensional velocity field $\boldsymbol{\vartheta} = \vartheta_i \boldsymbol{e}_i$ is approximated by[1] (see Caulk and Naghdi (1987)):

$$\boldsymbol{\vartheta} = \boldsymbol{v} + \sum_{N=1}^{k} x_{\alpha_1} \ldots x_{\alpha_N} \boldsymbol{W}_{\alpha_1 \ldots \alpha_N}, \tag{1}$$

with

$$\boldsymbol{v} = v_i(z,t)\boldsymbol{e}_i, \quad \boldsymbol{W}_{\alpha_1 \ldots \alpha_N} = W^i_{\alpha_1 \ldots \alpha_N}(z,t)\boldsymbol{e}_i. \tag{2}$$

This velocity field approximation (1) satisfies both the incompressibility condition and the kinematic boundary condition exactly. In condition (1), $\boldsymbol{v}$ represents the velocity along the axis of symmetry z at time $t, x_{\alpha_1} \ldots x_{\alpha_N}$ are the polynomial weighting functions with order k (this number identifies the order of the hierarchical theory and is related to the number of directors), the vectors $\boldsymbol{W}_{\alpha_1 \ldots \alpha_N}$ are the director velocities which are symmetric with respect to their indices and $\boldsymbol{e}_i$ are the associated unit basis vectors. The selection of such weighting functions represents an important aspect of the formulation of our problem. A good choice of these weighting functions can reduce the complexity of the system of partial differential equations in the director formulation of the theory. This choice should be consistent with the hierarchical structure of the basic theory so that the equations for each level of the hierarchy include the equations of all lower orders. The vectors $\boldsymbol{W}_{\alpha_1 \ldots \alpha_N}$ are related to physical features of the fluid, in particular the swirling motion – also called rotational motion. Using this approach with nine directors (i.e., $k = 3$ at condition (1)) and integrating the equations for the conservation of linear momentum over a circular cross-section of the fluid domain, we obtain unsteady relations between mean pressure gradient, volume flow rate and the swirling scalar function, over a finite section of the tube. Furthermore, we obtain the corresponding unsteady equation for the wall shear stress, which enters directly in the formulation as a dependent variable, and the also a partial differential equation for the swirling scalar function. Some numerical simulations are provided for unsteady flow regimes in a constricted rigid tube.

2. Equations of motion

Let x_i be the rectangular cartesian coordinates system and $\boldsymbol{x} = (x_1, x_2, x_3)$ where, for convenience, we set $x_3 = z$. We consider the isothermal flow of an homogeneous fluid in a (three-dimensional) straight tube of variable circular cross-section Ω (see Fig. 1). Also, let us consider the surface scalar function $\phi(z,t)$, that is related with the straight tube of circular cross-section by the following relation

$$\phi^2(z,t) = x_1^2 + x_2^2. \tag{3}$$

The boundary $\partial\Omega$ consists in the inlet cross-section Γ_1, the outlet cross-section Γ_2 and the lateral wall of the tube, denoted by Γ_w. Considering the flow of an incompressible viscous fluid without body forces in Ω, the equations of motion, stating the conservation of linear momentum and mass are given, in $\Omega \times (0,T)$, by

[1] In the sequel, latin indices subscript take the values $1, 2, 3$; greek indices subscript $1, 2$, and the usual summation convention is employed over a repeated index.