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Three dimensions simulation for the problem of a layer of non-Boussinesq fluid heated internally with prescribed heat flux on the lower boundary and constant temperature upper surface

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ABSTRACT

The purpose of this paper is to study the effect of a heat source on the solution to the equations for an incompressible heat conducting viscous fluid. The validity of both the linear instability and global nonlinear energy stability thresholds are tested using three dimensional simulation. Our results show that the linear threshold accurately predicts on the onset of instability in the basic steady state. However, the required time to arrive at the steady state increases significantly as the Rayleigh number tends to the linear threshold.

1. Introduction

Rayleigh–Bénard problem is the major section for the problem of the onset of convection in a horizontal fluid layer uniformly heated from below. Rayleigh (1916) provided an analysis on the assumption that the convection was induced by buoyancy effects. Rayleigh introduced an approximation to the basic equations of motion that he ascribed to Boussinesq (1903). However, Joseph (1976) found that the approximation had been earlier applied by Oberbeck (1879). The parameter whose value determines the onset of convection is called the Rayleigh number. Joseph (1976) noticed that this parameter appeared in a study by Lorenz (1881), who also used the approximation employed by Oberbeck.

The Oberbeck–Boussinesq approximation is the basis of most of the contemporary studies on natural or mixed convection flows. In the Oberbeck–Boussinesq approximation, all fluid properties such as viscosity and density can be taken as constants except that a buoyancy term proportional to a density difference is retained in the momentum equation. Thus, the fluid is taken as quasi-incompressible, the divergence of the velocity is approximated by zero in the continuity equation, and the term involving the product of the pressure and the divergence of the velocity is neglected in the thermal energy equation; see, for example, Section 8 of Chandrasekhar (1961). Several Oberbeck–Boussinesq approximations have been applied on the full Navier–Stokes equations. It is generally used in the framework of the natural convection problems such as the Rayleigh–Bénard configuration, and provides a simplified set of equations which is much more tractable for both numerical and

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analytical purposes, since all the acoustic scales have been eliminated. Rayleigh (1916) employed the simplified thermal energy equation and he ascribed it to Boussinesq (1903). In Rajagopal et al. (1996), intend to provide a rigorous derivation of the Oberbeck–Boussinesq approximation in the framework of a full thermodynamical theory of the Navier–Stokes equations. Then, Rajagopal et al. (1998) introduced a justification for the approximation within the full thermodynamical theory for Navier–Stokes–Fourier fluids. Hills and Roberts (1991) provided important idea to adapt a new method of treating the constraint of mechanical incompressibility. Recently, the Oberbeck–Boussinesq approximation have been developed intensely by Rajagopal (2006), Rajagopala et al. (2009), Barletta (2009) and Barletta and Nield (2009, 2010).

Straughan (1991) obtained quantitative non-linear stability estimates which guarantee nonlinear stability for the problem of penetrative convection in a plane layer with a nonuniform heat source, and a constant temperature upper surface, while the lower surface is subject to a prescribed heat flux. In addition to the non-linear results which establish a critical Rayleigh number below which convection cannot occur, Straughan (1991) calculated the linear value above which convection occurs. When the difference between the linear and nonlinear thresholds is very large, the comparison between these thresholds is very interesting and useful. Thus we repeat the stability analysis of Straughan (1991) to select new situations which have very big subcritical region. Then, we develop a three dimensions simulation for the problem. To do this, firstly, we transform the problem to velocity–vorticity formulation, then we use a second order finite difference schemes. We use implicit and explicit schemes to enforce the free divergence equation. The size of the Box is evaluated according to the normal modes representation. Moreover, we adopt the periodic boundary conditions for velocity, temperature, and concentration in the x, y dimensions.

In the next Section we present the governing equations of motion and derive the associated perturbation equations and then in Section 3, we introduce the linear and nonlinear analysis of our system. In Section 4, we transform our system to velocity–vorticity formulation. Section 5 is devoted to a study numerical solution of the problem in three dimensions. The results of our numerical investigation are then compiled and discussed in the final Section of the paper.

2. Governing equations

Consider then a layer of heat-conducting viscous fluid with a quadratic equation of state, occupying the horizontal layer $z \in (0, d)$ with the lower boundary $z = 0$ heated by radiation and with the temperature scale selected so that the temperature at $z = d$ remains a constant, T_u. By assuming the validity of the Oberbeck–Boussinesq approximation, the following local balance equations hold:

$$v_{i,t} + v_j\, v_{i,j} = -\frac{1}{\rho_m} p_{,i} + \nu\, \Delta v_i - g k_i [1 - \alpha (T - T_m)^2], \tag{2.1}$$

$$v_{i,i} = 0, \tag{2.2}$$

$$T_{,t} + v_i T_{,i} = \kappa \Delta T + Q, \tag{2.3}$$

where v, p, T, ν, g, α, and κ are respectively velocity, pressure, temperature, viscosity, gravity, a thermal expansion coefficient, and thermal diffusivity, $k = (0, 0, 1)$, and standard indicial notation is employed. These equations are defined on the spatial region $\mathbb{R}^2 \times [0, d]$. Here, we have to mention that the effects of pressure work are not taken into account in the energy balance. The boundary conditions are

$$\mathbf{v} = 0, \quad \text{at } z = 0, d, \quad T = T_u, \quad \text{at} \quad z = d, \quad \frac{\partial T}{\partial z} = \gamma, \quad \text{at } z = 0. \tag{2.4}$$

We here consider the heat supply function as $Q = Q_0(e^{z/d} - 1)$, where Q_0 and γ are constants. The steady solution $(\bar{\mathbf{v}}, \overline{T})$ corresponding to boundary conditions (2.4) is

$$\bar{\mathbf{v}} = 0, \quad \overline{T} = T_u - \gamma(d - z) + \frac{Q_0 d^2}{\kappa}\left(e - \frac{3}{2} - e^{\frac{z}{d}} + \frac{z}{d} + \frac{z^2}{2d^2}\right),$$

the hydrostatic pressure being determined from the momentum equation.

To investigate the stability of these solutions, we introduce perturbations (u, θ, π) by

$$v_i = \bar{v}_i + u_i, \quad T = \overline{T} + \theta, \quad p = \bar{p} + \pi.$$

Then, the perturbation equations are nondimensionalized according to the scales (stars denote dimensionless quantities)

$$t = t^* \frac{d^2}{\nu}, \quad U = \frac{\nu}{d}, \quad x = x^* d, \quad \theta = \theta^* T^\sharp, \quad \delta = \frac{\kappa (T_m - T_u)}{Q_0 d^2},$$

$$Pr = \frac{\nu}{\kappa}, \quad T^\sharp = U\sqrt{\frac{\nu}{\alpha g d \kappa}}, \quad R^2 = \frac{Q_0^2 d^7 g \alpha}{\kappa^3 \nu}, \quad \hat{\gamma} = \frac{\kappa \gamma}{Q_0 d}.$$

Here Pr is the Prandtl number and R^2 is a Rayleigh number. The dimensionless perturbation equations are (after omitting all stars)

$$u_{i,\,t} + u_j\, u_{i,j} = -\pi_{,\,i} + \Delta u_i + 2R f_1(z) \theta k_i + Pr k_i \theta^2, \tag{2.5}$$