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Weakly nonlinear waves in fluids of low viscosity: Lagrangian and Eulerian descriptions

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Dedicated to Professor Michael Carroll in warm recognition of his friendship and many talents.

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ABSTRACT

Acoustics equations are derived in low-viscosity newtonian fluids, when nonlinear effects are of first order relative to a small dimensionless parameter ϵ, which is a measure of the Mach number. Another small dimensionless parameter ζ is used to define *low-viscosity* precisely. In this context, using conservation of mass and of linear momentum, one derives governing equations for complex motions (simultaneous forward and backward propagation) and simple motions (forward propagation only). Propagation equations are obtained for four physical quantities (particle displacement, particle velocity, mass density, and pressure) in eulerian as well as in lagrangian form. For simple waves, the equations for particle velocity, mass density and pressure are found to be of the Burgers type; that for the displacement is not of the Burgers type. Consistent with the weak nonlinearity, proper boundary conditions for the simple-wave equations are derived in both eulerian and lagrangian forms; these new results are expressed only in terms of the source displacement and the fluid constants.

1. Introduction

The purpose of this paper is to derive acoustics equations in low-viscosity newtonian fluids when nonlinear effects are weak. This model can be used, at least as a first approximation, to describe propagation in biological tissue with an important application in ultrasound imaging. Indeed, it is well-known that biological tissue do not behave as newtonian fluids, where the attenuation coefficients in the range of low viscosity are proportional to the square of the frequency. In contrast, attenuation coefficients in biological tissue are proportional to powers of the frequency around unity. At the same time, tissue are complex materials composed of 85% or more water. In view of this, and of the complexity of analytical viscoelastic models for tissue, it is accepted that imaging software based on newtonian models give results of good accuracy in medical practice.

The paper is also intended to add to the existing body of knowledge in acoustics, for reasons that are detailed in the following. First, we derive nonlinear wave equations for particle displacement, particle velocity, mass density and pressure, together with boundary conditions consistent with the nonlinear theory, some of which are new.

Second, we provide answers to two questions that are left open in Whitham's monograph. In the author's words (Whitham, 1999, p. 96)

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The simplest equation combining both nonlinear propagation effects and diffusivity effects is Burgers' equation

$$c_t + c\ c_x = \nu\ c_{xx}. \tag{1.1}$$

... In general, if the two effects are important in a problem, there is usually some way of extracting (1.1) either as a precise approximation or as a useful basis for rough estimates.

In Eq. (1.1), c is the propagation speed, the indices attached to c denote partial derivatives, and ν is a constant. After making the statement quoted above, Whitham discusses at length properties and solutions of (1.1). He leaves two questions open, however. The first one concerns the method of approximation that leads to (1.1) in acoustics, for which no indications (or references) are given. The second one is in relation to the constant ν, which remains unspecified in terms of the shear and compression viscosities.

Third, we consider the derivation of simple-wave equations (forward propagation only) from complex-wave equations (simultaneous forward and backward propagation). To illustrate the point, consider the *classical* wave equation

$$\frac{\partial^2 f}{\partial t^2} + c^2\,\frac{\partial^2 f}{\partial x^2} = 0, \tag{1.2}$$

where $f(x,t)$ represents a function of the space variable x and time t. Solutions of this equation take the form of both forward and backward waves propagating with speed c. Such equations in acoustics are derived directly from first principles, namely conservation of mass and conservation of linear momentum, together with some appropriate approximations. On the other hand, consider the *simple-wave* equation

$$\frac{\partial f}{\partial t} + c\,\frac{\partial f}{\partial x} = 0, \tag{1.3}$$

which, as solutions, admits only of forward waves propagating with speed c. Eq. (1.3) does not in general follow trivially from (1.2), except when the speed c is a constant and it suffices to split the differential operator of (1.2) into two to justify the validity of (1.3).

The situation becomes more complicated when c in (1.2) is not a constant, as for example when c is a function of f and (1.2) becomes nonlinear. To deal with this difficulty, one approach is that of Hamilton and Blackstock (1998) (pp. 50–57). These authors transform equation (41) (a complex-wave equation) into Burgers equation (54) (a simple-wave equation). This, however, involves several questionable steps: (i) discarding the term $\mathcal{L}$ in their (41); (ii) introducing an auxiliary variable $\tilde{p}$ and then assuming that $\tilde{p} = p$; (iii) introducing a retarded time frame and slow scale; (iv) and finally removing their parameter $\tilde{\epsilon}$ in order to return to the physical coordinate x. To do away with those steps, we propose here a more direct approach to derive Burgers equation for the pressure, as in Hamilton and Blackstock (1998), but also extend the work to particle velocity and mass density.

Fourth, we examine the assumptions that lead to second-order approximation theory. From Hamilton and Blackstock (1998) (pp. 73–74)

Two assumptions underlie second-order approximation theory. First, the waves are not exceedingly strong; ... Second, distortion is dominated by cumulative effects; ... The immediate consequences of these two assumptions are as follows:

...

2. *Finite displacement of a source from its rest position may be ignored ...*
3. *The difference between material (Lagrangian) and spatial (Eulerian) representations may be ignored ...*

Boundary conditions (at the source), however, should be consistent with the second-order approximation, which makes their consequences 2 and 3 questionable. To elaborate, asssume that the source is a piston-like device and the fluid remains in contact with the device as it vibrates, so that it is permissible to identify the source displacement with that of the particles in contact with it. The resulting boundary condition is easily expressed in the lagrangian representation. In the eulerian representation, however, the boundary condition should not be written as though the source was not moving (see Hamilton & Blackstock, 1998, Eq. (3), p. 68). More on this subject can be found in the last section of the paper, where we derive boundary conditions that take into account second-order terms for all four physical quantities (particle displacement, particle velocity, mass density and pressure). The additional second-order terms make it clear that the difference between lagrangian and eulerian representations cannot be ignored. Indeed, even though the equations for simple waves *look* identical in the two representations (as can be seen in the last section), nonidentical boundary conditions will yield distinct results. In addition, there is another reason to keep from identifying the two representations: complex-wave equations do not *look* identical, as we shall see later in the last section.

To place this work in its context, we recall that nonlinear wave propagation has a long history, stretching back to over two hundred years. Originally, the field developed when rational explanations were sought to natural phenomena such as finite-amplitude waves in gas dynamics or solitary waves in open channels. Illuminating discussions, as well as historical perspectives, on nonlinear acoustics can be found in Whitham (1999), Hamilton and Blackstock (1998) and Drazin and Johnson (1989). We record here that the method of characteristics, which is the object of a thorough presentation in Whitham (1999), proved a tool of major importance in the development of nonlinear wave analysis. Among important works in nonlinear acoustics we may cite (Coulouvrat, 1991, 1992; Khokhlov & Soluyan, 1964; Rudenko & Soluyan, 1977; Mendousse,