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An upper bound for the steady-state temperature for a class of heat conduction problems wherein the thermal conductivity is temperature dependent

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ABSTRACT

This article presents an a priori upper bound estimate for the steady-state temperature distribution in a body with a temperature-dependent thermal conductivity. The discussion is carried out assuming linear boundary conditions (Newton law of cooling) and a piecewise constant thermal conductivity (when regarded as a function of the temperature). These estimates consist of a powerful tool that may circumvent an expensive numerical simulation of a nonlinear heat transfer problem, whenever it suffices to know the highest temperature value. In these cases the methodology proposed in this work is more effective than the usual approximations that assume thermal conductivities and heat sources as constants.

1. Introduction

Conduction heat transfer problems are usually simulated assuming temperature independent thermal conductivity. Such approximation strongly simplifies the simulation of the considered problems. However, temperature-dependent thermal conductivity is present in many problems with engineering relevance. For instance, concerning carbon nanotubes (known for their high thermal conductivities), the temperature-dependent thermal conductivity of crystalline ropes of single-walled carbon nanotubes decreases smoothly with decreasing temperature, and displays a linear temperature-dependence below 30 K (Hone, Whitney, Piskoti, & Zettl, 1999). Osman and Srivastava (2001) analyzed the temperature-dependent thermal conductivity of single-wall carbon nanotubes observing a peaking behavior in the thermal conductivity, as a function of temperature, before falling off at higher temperatures, using molecular dynamics simulations with the Tersoff–Brenner bond order potential. Zain-ul-Abdein, Azeem, and Shah (2012) studied the thermal conductivity in a particulate filled composite, verifying its dependence on the particle size, which, in turn, depends on the temperature.

Kim (2001) proposed a direct method to estimate the temperature-dependent thermal conductivity, employing the Kirchhoff transformation, which transforms the steady-state nonlinear heat conduction equation without heat source into the Laplace equation. The thermal conductivity, expressed as a linear combination of known functions with unknown coefficients, is determined from the imposed heat flux and the (measured) temperatures at the boundaries. Okoya and Ajadi (1999) studied thermal stability considering the thermal conductivity as exponential and power-law functions of temperature, in order to define the conditions for explosion, i.e. how explosion is affected by boundary conditions and how the

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Nomenclature

inf	denotes the infimum
max	denotes the maximum
sup	denotes the supremum
$\hat{f}(T)$	Kirchhoff transformation
$\hat{f}^{-1}(\omega)$	inverse of the Kirchhoff transformation
h	convection heat transfer coefficient
k	thermal conductivity
$\mathbf{n}$	unit outward normal
$\dot{q}$	internal heat source
$\mathbf{q}$	heat flux
R	radius
r	radial variable
T	temperature
T_0	reference temperature
T_∞	temperature of the environment
Γ	spatial region
$\partial\Gamma$	boundary of Γ
Ω	spatial region
$\partial\Omega$	boundary of Ω
$\partial\Omega^+$	subset of $\partial\Omega$
Ω^*	spatial region containing Ω
ω	function obtained from the Kirchhoff transformation

critical parameter is affected by a given constant. Moitsheki, Hayat, and Malik (2010) present exact solutions of a nonlinear fin problem when both thermal conductivity and heat transfer coefficient are power law functions of the temperature, by employing classical Lie symmetry techniques.

Temperature-dependent thermal conductivity plays an important role in porous silicon, especially regarding its applications in optoelectronics. Geseley, Linsmeieryx, Drachy, Frickey, and Arens-Fischerz (1997) verified that the thermal conductivity increases with temperature increase. Another important material presenting interest in optoelectronic and electronic applications is zinc oxide. The maximum thermal conductivities of the polycrystalline zinc oxide occur at about 60 K and their values are almost an order of magnitude lower than bulk ZnO (Alvarez-Quintana, Martınez, Pérez-Tijerina, Pérez-García, & Rodríguez-Viejo, 2010).

Nevertheless, even taking into account the dependence of the thermal conductivity on the temperature, it is possible to estimate a priori an upper bound for the temperature field, without the needing of a complete simulation of the conduction heat transfer process. Sometimes the simulation of a complex nonlinear heat transfer problem is carried out only for verifying if the maximum temperature remains lower than a given bound. In such cases the simulation is no longer required if an upper bound estimate for the solution is already available. In particular, when the thermal conductivity k is approximated by a piecewise constant function of the temperature as follows

$$k = \begin{cases} k_1 = \text{constant} & \text{for } T_0 < T \\ k_2 = \text{constant} & \text{for } T_0 \geqslant T \end{cases} \tag{1}$$

in which T_0 is a constant, the upper bound estimate becomes especially easy to be obtained. Eq. (1) represents a first approximation for problems with temperature-dependent thermal conductivity.

The main objective of this work is to provide an a priori upper bound estimate for the temperature distribution in homogeneous bodies with piecewise constant temperature-dependent thermal conductivity, subjected to a linear boundary condition (Newton's law of cooling).

This a priori estimate may be useful, for instance, when the main goal is to ensure that a (maximum admissible) temperature will not be reached. It is important to note that upper bounds for problems subjected to nonlinear boundary conditions (conduction/radiation heat transfer) for constant thermal conductivity have already been proposed (Saldanha da Gama, 1997, 2000).

The classical steady-state conduction heat transfer process in a rigid and opaque body at rest, represented by the bounded open set Ω with boundary $\partial\Omega$, subjected to a linear boundary condition is mathematically described by (Incropera & Dewitt, 1996)

$$\begin{aligned} &div[k\, gradT] + \dot{q} = 0 \quad \text{in} \quad \Omega \\ &-k\, gradT \cdot \mathbf{n} = h(T - T_\infty) \quad \text{on} \quad \partial\Omega \end{aligned} \tag{2}$$