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An approach to generalized one-dimensional self-similar elasticity

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ABSTRACT

We employ a self-similar Laplacian in the one-dimensional infinite space and deduce a model for one-dimensional self-similar elasticity. As a consequence of self-similarity this Laplacian assumes the non-local form of a self-adjoint combination of fractional integrals. The linear elastic constitutive law becomes a non-local convolution with the elastic modulus function being a power-law kernel. We outline some principal features of a linear self-similar elasticity theory in one dimension. We find an anomalous behavior of the elastic modulus function reflecting a regime of critically slowly decreasing interparticle interactions in one dimension. The approach can be generalized to the n $(n = 1, 2, 3)$ dimensional physical space (Michelitsch, Maugin, Nowakowski, Nicolleau, & Rahman, to be published).
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1. Introduction and prerequisites

Emmy Noether discovered in 1915 the fundamental link between symmetries and conservation laws in nature. This fundamental discovery of physics is published in a celebrated paper Noether (1918). In 1921 Bessel-Hagen derived 15 conservation laws based on Noether's theorem (Bessel-Hagen, 1921) where not all of them seem to have any physical relevance. However, it was much later, namely in the seventies of the last century, that *scaling invariances* attracted the attention of scientists first in the context of phase transitions and critical phenomena followed by the great discoveries of *Fractal Geometry* by Mandelbrot (1983, 1984, 1997), Sapoval (1997) and of the universal laws underlying deterministic chaos established by Feigenbaum (1978) and others.

The goal of the present paper is to analyze a one-dimensional continuous medium with self-similar (scaling invariant) harmonic interparticle interactions (interactions between mass points).

The starting point of the model is a self-similar elastic energy expression which is used to deduce the self-similar stress–strain constitutive relation. This kind of self-similarity causes inevitably non-locality of the linear constitutive relations in the form of fractional derivatives. There is an increasing interest in fractional approaches in engineering sciences and physics as they open the possibility to model complex material behavior (e.g. Drapaca & Sivaloganathan, 2010 and the references therein) and to take into account scaling behavior and nonlocality in continuum mechanics (Eringen, 2002).

In the present paper we utilize a continuous representation of a self-similar, and as a consequence, non-local Laplacian involving fractional derivatives which we deduced earlier (Michelitsch, 2011; Michelitsch et al., 2012). With this framework

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a 1D continuum is defined where the scaling invariance is the intrinsic symmetry of the elastic properties. We can hence call this medium a "self-similar medium" or synonymously a "scaling invariant medium", namely by characterizing the symmetry of the elastic modulus tensor. This paper is devoted to the linear self-similar elasticity in one dimension. The exponents which appear in the elastic material function kernels define characteristic regimes of the self-similar medium in one dimension. We emphasize that the term "self-similar medium" is used here to describe the self-similarity (scaling invariance) of the *interparticle interactions*. Self-similar or fractal spatial mass distributions of particles are not considered. We refer in that context to the works of Ostoja-Starzewski (2009) and the references therein considering the continuum mechanics of fractal mass distributions. Despite we restrict ourselves to a 1D medium, the present approach can be extended to continuum field theories in 2D, 3D and higher dimensions (Michelitsch et al., to be published). In 1D we applied the present field approach to model wave propagation (Michelitsch et al., 2012) and anomalous diffusion (Michelitsch, 2011; Michelitsch et al., 2012) in media with self-similar interparticle interactions.

First of all let us introduce some basic notions and prerequisites and define the notion of *self-similarity* which we are utilizing. We call a function $\Lambda(h)$ *self-similar* with respect to a continuous variable $h > 0$ at $h = 0$ if the property holds

$$\Lambda(Nh) = N^{\delta}\Lambda(h) \tag{1}$$

for a certain prescribed scaling factor $N \in \mathrm{R}_+$ and where δ denotes a scaling dimension. Relation (1) implies for a fix N that $\Lambda(N^s h) = N^{\delta s}\Lambda(h)$ for all positive and negative integers $s \in \mathrm{Z}_0$ including the zero. An equivalent definition of self-similarity is obtained when we consider the function $h^{-\delta}\Lambda(h) = \tilde{\Lambda}(\ln h)$ which is invariant under the transformation $h' = N^s h$ and is hence a $\ln N$- periodic function of the variable $\ln(h)$ and as a consequence it holds that $\tilde{\Lambda}(\ln h + s\ln N) = \tilde{\Lambda}(\ln h)$ for any $s \in \mathrm{Z}_0$.

The above introduced notion of self-similarity corresponds to the notion of "self-similarity at a point" used in the fractal mathematical literature (Peitgen, Jürgens, & Saupe, 1991). The simplest kind of self-similar functions with respect to h is provided by power-functions h^{δ}.

Before we begin to specify the physical model of 1D self-similar continua it is convenient to give some mathematical prerequisites which will be needed in the physical model. To this end we consider the Fourier transform of $|k|^{\alpha}$ defined by

$$b_{\alpha}(x) = \lim_{\epsilon\to 0+}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-|k|\epsilon}|k|^{\alpha}\,\mathrm{d}k = \lim_{\epsilon\to 0+}\frac{1}{\pi}\int_{0}^{\infty} e^{-\epsilon k}k^{\alpha}\cos kx\,\mathrm{d}k, \quad \alpha > -1 \in \mathbb{R} \tag{2}$$

which exists only for $\alpha > -1$ and where $\lim_{\epsilon\to 0+}(..)$ means ϵ approaches zero from the positive side. In order to define (2) it is convenient to add the factor $e^{-\epsilon k}$ and consider the integral in the limit $\epsilon \to 0+$. Since $e^{-\epsilon k} \to 1$ for any finite k this factor is hence only important "under the integral". For $\alpha \leqslant -1$ this integral diverges at $k = 0$. The divergent case $\alpha < -1$ of (2) requires a regularization procedure in order to become well-defined. This regularization will be outlined below in Section 1.2. We consider first (2) for the regular case $\alpha > -1$. Then it will be convenient to elaborate briefly a regularization technique which is based on the concept of generalized functions or distributions in the spirit of Gel'fand and Shilov (1964).

1.1. Prerequisite 1: Fourier transform of $|k|^{\alpha}$ (2) for $\alpha > -1$

By introducing the new variable $s = k(\epsilon - i|x|)$ we can write

$$b_{\alpha}(x) = Re\left\{\lim_{\epsilon\to 0+}\frac{1}{\pi}\frac{1}{(\epsilon - ix)^{\alpha+1}}\int_0^{\infty} e^{-s}s^{\alpha}\,ds\right\} = \frac{\alpha!}{\pi}Re\left\{\lim_{\epsilon\to 0+}\frac{i^{\alpha+1}}{(x+i\epsilon)^{\alpha+1}}\right\}, \quad \alpha > -1 \in \mathbb{R} \tag{3}$$

We have introduced in (3) the Γ-function (faculty function) $\alpha! = \Gamma(\alpha+1)$ which is defined for complex arguments z in the form (Abramowitz & Stegun, 1972)

$$\Gamma(z+1) =: z! = \int_0^{\infty} e^{-\tau}\tau^{z}\,\mathrm{d}\tau, \quad Re(z) > -1 \tag{4}$$

which exists for $Re(z) > -1$. We note that $\frac{i^{\alpha+1}}{(x+i\epsilon)^{\alpha+1}}$ in (3) assumes the complex conjugate value under replacement $x \leftrightarrow -x$. So its real-part depends only on $|x|$ and we can write

$$b_{\alpha}(x) = b_{\alpha}(|x|) = \frac{\alpha!}{\pi}Re\left\{\lim_{\epsilon\to 0+}\frac{i^{\alpha+1}}{(|x|+i\epsilon)^{\alpha+1}}\right\}, \quad \alpha > -1 \in \mathbb{R} \tag{5}$$

For $x \neq 0$ we obtain

$$b_{\alpha}(x) = -\frac{\alpha!}{\pi|x|^{\alpha+1}}\sin\left(\frac{\alpha\pi}{2}\right) \tag{6}$$

Relation (6) is in accordance with the result given by Gel'fand and Shilov for the Fourier transform of $|k|^{\alpha}$ (p. 447, Eq. 16 in Gel'fand & Shilov, 1964).

We note the following observations:

(i) $b_{\alpha}(x) = b_{\alpha}(-x) = b_{\alpha}(|x|)$ is a symmetric function in x.

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