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VAM applied to dimensional reduction of non-linear hyperelastic plates

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ABSTRACT

This work aims at dimensional reduction of non-linear isotropic hyperelastic plates in an asymptotically accurate manner. The problem is both geometrically and materially non-linear. The geometric non-linearity is handled by allowing for finite deformations and generalized warping while the material non-linearity is incorporated through hyperelastic material model. The development, based on the Variational Asymptotic Method (VAM) with moderate strains and very small thickness to shortest wavelength of the deformation along the plate reference surface as small parameters, begins with three-dimensional (3-D) non-linear elasticity and mathematically splits the analysis into a one-dimensional (1-D) through-the-thickness analysis and a two-dimensional (2-D) plate analysis. Major contributions of this paper are derivation of closed-form analytical expressions for warping functions and stiffness coefficients and a set of recovery relations to express approximately the 3-D displacement, strain and stress fields. Consistent with the 2-D non-linear constitutive laws, 2-D plate theory and corresponding finite element program have been developed. Validation of present theory is carried out with a standard test case and the results match well. Distributions of 3-D results are provided for another test case.

1. Introduction

Non-linear isotropic hyperelastic materials have potential applications in space based inflatable structures, pneumatic membranes, replacements for soft biological tissues, prosthetic devices, compliant robots, high altitude airships and artificial blood pumps, to name a few. They have special engineering properties like high strength-to-mass ratio, low deflated volume and low density (Jenkins, 2001). They are subjected to large strains and large deformations (and rotations) due to externally applied loads.

Despite their potential applications and special engineering properties, there are no generalized analytical and numerical model characterizations as an alternative to experimental results. The first significant theoretical study of membranes was carried out by Adkins and Rivlin (1952), using the neo-Hookean (Rivlin, 1948) and Mooney forms (Mooney, 1940) of strain energy function. A summary of this work was given by Green and Adkin (1960). Later in the 1960s, Hart-Smith and Crisp (1967) and Klingbeil and Shield (1964) examined the special case of axisymmetrical hyperelastic membrane's inflation. These authors proposed analytical solutions for the circular plane membrane inflation problems using different hyperelastic non-linear constitutive equations. Further, Hart-Smith and Crisp (1967) used thickness-wise distribution of deformation to estimate the deviation in profile from spherical shape. It is also known that qualitative differences exist between the behavior of bodies made of compressible elastic materials and incompressible elastic materials under the same boundary conditions (Adkins & Rivlin, 1955; Chaudhry & Singh, 1969; Chien-Heng & Widera, 1969; Kerr & Tang, 1967). Mathematical

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formulations for compressible materials are considerably more complicated than for incompressible materials (see Beatty (1987), Carroll (1988), Carroll & Horgan (1990), Jiang & Ogden (2000) and references listed in Zidi & Cheref (2002)). For very large deformation, asymptotic solutions are possible, greatly simplifying the analysis (see Ferner & Wu, 1981; Wang & Shield, 1969). An asymptotic expansion is provided for large elastic strain of a circular plate made of isotropic, incompressible material by Taber (1987), but the model ignores transverse shear deformation. An asymptotic theory for thin, compressible hyperelastic plates is derived by Erbay and Suhubi (1991). They assume that the displacement vector and the stress tensor are expandable into asymptotic series. Further, Erbay (1997) derived asymptotic non-linear membrane theory of compressible and incompressible hyperelastic plates for general hyperelastic constitutive models and the effect of material non-linearity. Application of finite element method to membrane problems was first carried out by Oden and Sato (1967). They considered three-noded triangular elements and the corresponding non-linear governing equations are solved using the Newton–Raphson algorithm for inflation of a plane circular membrane. State of the art developments in membrane analysis are reviewed by Jenkins and Leonard (1991), and Jenkins (1996). Current developments of plate as 2-D and 3-D elements with various kinematic models, constitutive forms and numerical integration techniques as well as their relative advantages and disadvantages are extensively reviewed in Yang, Saigal, and Liaw (1990), Yang, Saigal, Masud, and Kapania (2000), Sze (2002) and references therein.

In the current analysis VAM has been applied to non-linear isotropic hyperelastic material model, thus the original three-dimensional (3-D) non-linear elastic problem splits into a non-linear one-dimensional (1-D), through-the-thickness analysis and a non-linear, two-dimensional (2-D) plate analysis. This greatly reduces the computational cost compared to 3-D non-linear finite element analysis. Through-the-thickness analysis provides a 2-D non-linear constitutive law for the plate equations and a set of recovery relations that express the 3-D field variables (displacements, strains and stresses) through-the-thickness in terms of 2-D plate variables calculated in the plate analysis (2-D). A unified software package 'VAMNLM' (Variational Asymptotic Method applied to Non-linear Material models) was developed to carry out 1-D non-linear analysis (analytical), 2-D non-linear finite element analysis and 3-D recovery analysis. Analytical expressions (asymptotically accurate) are derived for stiffness, strains, stresses and 3-D warping field. Validation of present theory is carried out with a standard test case. Preliminary forms of current analysis results match well with the literature. This paper is the *first* to provide asymptotically correct dimensional reduction approach to accurately and efficiently reproduce the complete 3-D results through-the-thickness for plate structures with nonlinear material model using VAM. To illustrate this novelty, one more test case is considered to provide complete description of the structural behavior in the form of 3-D displacements, strains (Green strains) and stresses (Cauchy and second Piola–Kirchhoff stresses) through-the-thickness of the plate.

2. Three-dimensional formulation

A plate, like any other physical structure, is a three-dimensional (3-D) continuum in reality, but because of its possession of a relatively small thickness h with respect to its other two dimensions, it can be represented as a two-dimensional (2-D) smooth reference surface usually chosen to be the mid-surface, in its undeformed state, mathematically represented by a set of two arbitrary, but independent curvilinear coordinates, x_α. Here and throughout the formulation, Greek indices assume values 1 and 2 while Latin indices assume 1, 2, and 3. Dummy indices are summed over their range, except where explicitly indicated. x_α are thus the surface coordinates while x_3 is the normal coordinate. Without loss of generality, lines of curvatures are chosen to be the curvilinear coordinate curves to simplify the formulation.

Fig. 1 provides a schematic of plate deformation. Let $\mathbf{b}_i$ and $\mathbf{B}_i$ denote the unit vectors along x_i in the undeformed and deformed configurations, respectively. One could then describe the positions of any material point $Q(x_1,x_2,x_3)$ in the undeformed and deformed configurations by its position vectors, $\hat{\mathbf{r}}$ and $\widehat{\mathbf{R}}$, respectively, from any point O, which is fixed in a reference frame whose motion itself is inertial and/or known, such that

$$\hat{\mathbf{r}}(x_1,x_2,x_3) = \mathbf{r}(x_1,x_2) + x_3\mathbf{b}_3(x_1,x_2), \tag{1}$$

$$\widehat{\mathbf{R}}(x_1,x_2,x_3) = \mathbf{R}(x_1,x_2) + x_3\mathbf{B}_3(x_1,x_2) + w_i(x_1,x_2,x_3)\mathbf{B}_i(x_1,x_2). \tag{2}$$

$w_i(x_1,x_2,x_3)$ in Eq. (2) above are components of 3-D warping field. Amongst them, w_1 and w_2 are in-plane warpings (due to local rotations of line elements normal to the reference surface) and w_3 is out-of-plane warping (stretching or contraction of the normal line elements). Thus the formulation accounts for the contraction or extension of the normal through-the-thickness. The covariant and contravariant base vectors in the undeformed state are, $\mathbf{g}_i = \frac{\partial \hat{\mathbf{r}}}{\partial x_i}$, $\mathbf{g}^i = \frac{1}{2\sqrt{g}} e_{ijk}\mathbf{g}_j \times \mathbf{g}_k$, respectively, where g is determinant of the metric tensor for the undeformed configuration, $g = \det(\mathbf{g}_i.\mathbf{g}_j)$ and e_{ijk} are components of the permutation tensor. Similarly, the covariant base vectors for the deformed configuration are given by $\mathbf{G}_i = \frac{\partial \hat{\mathbf{R}}}{\partial x_i}$. The relation between $\mathbf{b}_i$ and $\mathbf{B}_j$ can be prescribed by an arbitrarily large rotation specified in terms of the matrix of direction cosines $\overline{C}(x_1,x_2)$ so that $\mathbf{B}_i = \overline{C}_{ij}\mathbf{b}_j$, $\overline{C}_{ij} = \mathbf{B}_i.\mathbf{b}_j$. In the present scheme, all possible deformations (large displacements and rotations) are allowed and the corresponding Green strain ($\mathbf{\Gamma}$) whose Lagrangian components (Crisfield, 2000) are

$$\Gamma_{ij} = \frac{1}{2}(F_{ki}F_{kj} - I_{ij}), \tag{3}$$

where F_{ij} are mixed bases components of the deformation gradient tensor (DGT), given by $F_{ij} = \mathbf{B}_i.\mathbf{G}_k\mathbf{g}^k.\mathbf{b}_j$ and I_{ij} are elements of the 3×3 identity matrix.