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Generalisation of elastic models for a layer with elastically restrained boundaries

Rinat R. Mukhomodyarov, Graham A. Rogerson *

School of Computing and Mathematics, Keele University, Keele, Staffordshire ST5 5BG, UK

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ABSTRACT

A series of long wave asymptotic models for an isotropic layer are developed for the case of elastically restrained boundary conditions (ERBC). The dispersion relation and cut-off frequency equation are established for the case of symmetric boundary conditions restrained in the normal and tangential directions. A long-wave low frequency asymptotic model is developed to describe motion associated with the fundamental modes for small values of the restraint parameters. Four high frequency approximate models are developed which describe all possible asymptotic regimes connected with vibration within the vicinity of thickness resonances. One of the features of these models is that the cut-off frequencies are given implicitly for all families of the frequency spectrum. A special uniform asymptotic model is developed for the case of coalescing long wave limits which interact with each other. In this case it is not possible to use any of the typical techniques associated with the classic asymptotic models.

1. Introduction

The development of the approximate theories describing the dynamic response of layered structures is based mostly on the assumption that a typical wavelength is large with respect to layer thickness. Following the terminology of Kaplunov, Kossovich, and Nolde (1998), we shall refer to this type of asymptotic theory as "long-wave". This class comprises not only the well-known classical low frequency theories of plates and shells, see for example Ciarlet (1997) or Goldenveizer (1961), but also approximate theories describing high frequency vibrations within the vicinity of thickness resonances, see for example Mindlin (2006), Achenbach (1969) or Berdichevsky (2009). An alternative approach, based on direct asymptotic integration of the exact boundary value problems of linear elasticity, was developed in Kaplunov et al. (1998) and later extended to include the effects of anisotropy (Kaplunov, Kossovich, & Rogerson, 2000), pre-stress (Pichugin & Rogerson, 2001) and kinematic constraints (Kossovitch, Moukhomodiarov, & Rogerson, 2002).

The majority of approximate models have been developed assuming the most commonly used traction free boundary condition on the outer surfaces, with the fixed face condition also receiving some limited attention, see Kaplunov (1995). In reality, there are many scenarios when layered structures interact with surrounding media, making the construction of appropriate boundary conditions relevant. Such boundary conditions may be regarded as providing a smooth transition between two limiting cases, namely traction free and fixed face boundary conditions. As a first approximation, it can be assumed that the field at the layer's boundaries satisfies a Hooke-type law, i.e. that the boundaries are elastically restrained. This approach has been used, for example, by Mindlin (1960), who considered wave propagation in an isotropic elastic layer with boundaries elastically restrained in the normal direction. Mindlin's study was recently extended to investigate more

* Corresponding author.
E-mail addresses: rinat2005@gmail.com (R.R. Mukhomodyarov), g.a.rogerson@maths.keele.ac.uk (G.A. Rogerson).

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general aspects of wave propagation in layers with elastically restrained boundaries, with particular emphasis on long-wave propagation and derivation of various asymptotic approximations of the dispersion relation, see Moukhomodiarov, Pichugin, and Rogerson (2010) and Mukhomodyarov and Rogerson (submitted for publication).

The aim of this present paper is to develop a series of long-wave asymptotic models for an isotropic layer subject to symmetric elastically restrained boundary conditions. The method of direct asymptotic integration of the boundary value problems of linear elasticity, under the assumption of plain strain, will be applied. Special attention will be paid to some aspects of the integration process arising specifically from the elastically restrained boundary conditions. One of these aspects is a mutual mode transition related to the restraint parameters which leads to non-uniformity of standard asymptotic models. This problem is resolved by constructing a special approximate model which is uniform in the vicinity of interacting long-wave limits.

The paper is organised as follows. In Section 2 we first formulate a statement of the problem for an isotropic layer with elastically restrained boundaries. A form of the corresponding dispersion relation and the frequency equation is then established before a short description of mode transition is presented. In Section 3, low frequency models are developed by using the asymptotic integration method for both the flexural and extensional motion. Generalisation of the classical models for the case of elastically restrained boundaries is established and analogy with the classical equations is discussed. In Section 4, the corresponding high frequency models, describing motion in the vicinity of thickness resonances, are derived. Their consistency is confirmed by comparison of the corresponding dispersion relations with long wave asymptotics of the exact dispersion relation. The special case of interacting long-wave limits is discussed in Section 5 for which the adopted method of asymptotic integration is elaborated to take into account a complex asymptotic regime taking place for modes having similar cut-off frequencies. The efficiency of the special model developed is shown by comparison with the general high frequency model.

2. Statement of the problem and dispersion relations

Consider an infinite layer of thickness $2h$ and composed of an isotropic linearly elastic material. The surfaces of the layer, defined relative to the Cartesian coordinate system $Ox_1x_2x_3$ as $|x_2| = h$, are assumed elastically restrained in both the normal and tangential directions. In the case of symmetry, these may be described by the following boundary conditions

$$\boldsymbol{\alpha}^{(1)}\boldsymbol{\tau} \pm \boldsymbol{\beta}^{(1)}\boldsymbol{u} = 0, \quad \boldsymbol{\alpha}^{(2)}\boldsymbol{\tau} \pm \boldsymbol{\beta}^{(2)}\boldsymbol{u} = 0 \quad \text{at} \quad x_2 = \pm h. \tag{1}$$

We restrict ourselves with the case within which the stresses are linked with the corresponding displacements independently, i.e. at $x_2 = \pm h$

$$\frac{1}{\mu}\left(\alpha_1\tau_1 \pm \beta_1\frac{\mu}{h}u_1\right) = \alpha_1\left\{\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right\} \pm \frac{\beta_1}{h}u_1 = 0, \tag{2}$$

$$\frac{1}{\rho}\left(\alpha_2\tau_2 \pm \beta_2\frac{\mu}{h}u_2\right) = \alpha_2\left\{(c_1^2 - 2c_2^2)\frac{\partial u_1}{\partial x_1} + c_1^2\frac{\partial u_2}{\partial x_2}\right\} \pm c_2^2\frac{\beta_2}{h}u_2 = 0, \tag{3}$$

where α_n, β_n, $n = 1, 2$, are non-dimensional positive scalars.

Within the plane strain assumption the motion of the layer is governed by the Navier equation

$$c_1^2\frac{\partial^2 u_1}{\partial x_1^2} + (c_1^2 - c_2^2)\frac{\partial^2 u_2}{\partial x_1 \partial x_2} + c_2^2\frac{\partial^2 u_1}{\partial x_2^2} = \frac{\partial^2 u_1}{\partial t^2}, \tag{4}$$

$$c_2^2\frac{\partial^2 u_2}{\partial x_1^2} + (c_1^2 - c_2^2)\frac{\partial^2 u_1}{\partial x_1 \partial x_2} + c_1^2\frac{\partial^2 u_2}{\partial x_2^2} = \frac{\partial^2 u_2}{\partial t^2}, \tag{5}$$

where $u_1 = u_1(x_1,x_2,t)$ and $u_2 = u_2(x_1,x_2,t)$ are the longitudinal and transverse displacements, respectively, $c_1 \equiv \sqrt{2(1-\nu)\mu/(1-2\nu)\rho}$ the dilatational wave speed, $c_2 \equiv \sqrt{\mu/\rho}$ the shear wave speed, ν Poisson's ratio, μ the shear modulus and ρ the mass density.

The dispersion relations for elastic wave propagation in a layer subject to the boundary conditions (2) and (3) were derived previously in Moukhomodiarov et al. (2010). These relations can be written in a compact form, for symmetric modes

$$F_1(\delta_1 q_1 \cosh\eta q_1 \cosh\eta q_2 + \delta_2 q_2 \sinh\eta q_1 \sinh\eta q_2) + F_2 \sinh\eta q_1 \cosh\eta q_2 + F_3 \cosh\eta q_1 \sinh\eta q_2 = 0 \tag{6}$$

and for antisymmetric modes

$$F_1(\delta_1 q_1 \sinh\eta q_1 \sinh\eta q_2 + \delta_2 q_2 \cosh\eta q_1 \cosh\eta q_2) + F_2 \cosh\eta q_1 \sinh\eta q_2 + F_3 \sinh\eta q_1 \cosh\eta q_2 = 0, \tag{7}$$

where

$$F_1 = \frac{1 - q_1^2}{\eta}, \quad F_2 = \frac{\delta_1\delta_2}{\eta^2} - (1 + q_1^2)^2, \quad F_3 = q_1 q_2\left(4 - \frac{\delta_1\delta_2}{\eta^2}\right),$$

within which $\eta \equiv kh$ is the non-dimension wave number, $\delta_i = \beta_i/\alpha_i$ (i = 1,2) and q_1, q_2 are given by