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A strain gradient functionally graded Euler–Bernoulli beam formulation

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ABSTRACT

A size-dependent functionally graded Euler–Bernoulli beam model is developed based on the strain gradient theory, a non-classical theory capable of capturing the size-effect in micro-scaled structures. The governing equation and both classical and non-classical boundary conditions are obtained using variational approach. To develop the new model, the previously used simplifying assumption which considered the length scale parameter to be constant through the thickness is avoided in this work. As a consequence, equivalent length scale parameters are introduced for functionally graded microbeams as functions of the constituents' length scale parameters. Moreover, a generally valid closed-form solution is derived for static deflection of the new model. As case studies, the static and free-vibration of the new model are investigated for FG simply supported microbeams in which the properties are varying through the thickness according to a power law and the results of the new model are compared to those of the modified couple stress and the classical continuum theories, noted that the two latter theories are special cases of the strain gradient theory utilized in this paper.

1. Introduction

Functionally graded materials (FGMs) are indeed a mixture of two different materials, typically a metal and a ceramic, with the advantages of both of their constituents. The properties of FGMs vary along certain dimension(s) following continuous functions. The continuity of the FGMs distribution functions is indeed a significant privilege that provides a solution to the high-shear-stress problems which appears in laminated composites, where two materials with great differences in properties are bonded. Today, structures made of FGMs play an important role in different industrial fields such as aerospace, biomechanics, etc. Some studies have been carried out by researchers on the static and dynamic behavior of FGM beams and plates. Some of these investigations are mentioned by Asghari, Ahmadian, Kahrobaiyan, and Rahaeifard (2010) and Asghari, Rahaeifard, Kahrobaiyan, and Ahmadian (2011) including Sankar (2001), Aydogdu and Taskin (2007), Ying, Lu, and Chen (2008), Xiang and Yang (2008), Kapuria, Bhattacharyya, and Kumar (2008), and Li (2008). All of the aforementioned studies are based on the classical continuum theory, while the current paper is formulated on the basis of a non-classical continuum theory, the strain gradient theory.

Recently, micro and nano structures made of functionally graded materials have found applications in shape memory alloys as thin films (Craciunescu & Wuttig, 2003; Fu, Du, & Zhang, 2003) and also in micro- and nano-electromechanical systems (MEMS and NEMS) (Fu, Du, Huang, Zhang, & Hu, 2004; Witvrouw & Mehta, 2005). The characteristic size (thickness, diameter, etc.) of beams used in MEMS and NEMS are in the order of microns and sub-microns; hence, the small-scale-effect in their behavior is significant. The experimental observations have indicated that the classical continuum mechanics is

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unable to predict or interpret the size-dependent static and vibration behavior observed in micro-scaled structures (Fleck, Muller, Ashby, & Hutchinson, 1994; Lam, Yang, Chong, Wang, & Tong, 2003; McFarland & Colton, 2005; Stolken & Evans, 1998); so, during past years, some non-classical continuum theories such as the nonlocal, strain gradient and couple stress theories have been introduced, developed and employed to investigate the micro-scaled structures.

The couple stress theory is a non-classical continuum theory in which higher-order stresses, known as the couple stresses exist (Asghari, Kahrobaiyan, Rahaeifard, & Ahmadian, 2011). A modified couple stress theory has been proposed by Yang, Chong, Lam, and Tong (2002) in which a new higher-order equilibrium equation, i.e. the equilibrium equation of moments of couples, is considered in addition to the classical equilibrium equations of forces and moments of forces. This theory has been employed to formulate the size-dependent static and dynamic behaviors of linear and nonlinear homogenous Euler–Bernoulli and Timoshenko beam models (Asghari, Kahrobaiyan, & Ahmadian, 2010; Kong, Zhou, Nie, & Wang, 2008; Ma, Gao, & Reddy, 2008; Park & Gao, 2006; Xia, Wang, & Yin, 2010) and linear homogenous Kirchhoff microplates (Jomehzadeh, Noori, & Saidi, 2011; Tsiatas, 2009). In addition, this non-classical theory is utilized to investigate the characteristics of some micro-scaled structures (Fu & Zhang, 2010; Gheshlaghi, Hasheminejad, & Abbasion, 2010; Kahrobaiyan, Asghari, Rahaeifard, & Ahmadian, 2010).

Considering the first and the second derivatives of the strain tensor, Mindlin (1965) proposed a higher-order gradient theory for elastic materials. Fleck and Hutchinson (1993, 1997, 2001) used Mindlin's formulations by considering only the first derivative of the strain tensor and called it the strain gradient theory. In comparison with the couple stress theory, the strain gradient theory contains some additional higher-order stress components beside the classical and couple stresses. Indeed, the couple stress theory is a special case of the strain gradient theory.

In a similar way utilized by Yang et al. (2002) for the modification of the couple stress theory, Lam et al. (2003) introduced a modified strain gradient theory, which reduces in a special case to the modified couple stress theory. Henceforth, when the strain gradient theory is used in the text, it denotes the version of the theory presented by Lam et al. (2003).

The strain gradient theory is employed to formulate the static and dynamic behaviors of linear Euler–Bernoulli and Timoshenko beam models respectively by Kong, Zhou, Nie, and Wang (2009) and Wang, Zhao, and Zhou (2010). They assessed the influence of the ratio of the beam thickness to the material length scale parameter on the static deflection and natural frequencies of microbeams. In addition, the size-dependent nonlinear Euler–Bernoulli beam model has been developed by Kahrobaiyan, Asghari, Rahaeifard, and Ahmadian (2011) based on the strain gradient theory. Furthermore, this non-classical theory is utilized by Akgoz and Civalek (2011) in order to study the buckling of microbeams. They compared the results of the strain gradient theory with the results of the modified couple stress and the classical theories. Also, torsion of strain gradient bars is formulated by Kahrobaiyan, Tajalli, Movahhedy, Akbari, and Ahmadian (2011). They indicated that the static and dynamic torsional behavior of microbars is size-dependent.

Not only homogenous, but also functionally graded beam models are formulated by researchers utilizing the non-classical theories. For example, employing the modified couple stress theory, linear FG Euler–Bernoulli and Timoshenko beam models are formulated by Asghari et al. (2010, 2011). In addition, a nonlinear modified couple stress FG Euler–Bernoulli beam model is developed by Ke, Wang, Yang, and Kitipornchai (2012). Furthermore, the dynamic stability of functionally graded modified couple stress Timoshenko microbeams is investigated by Ke and Wang (2011). It is noted that the reviewed works on the size-dependent behaviors of the FG beams are based on the modified couple stress theory. Where as, in this work, for the first time, the strain gradient theory modified by Lam et al. (2003) is employed to capture the size-effects of FG microbeams. The strain gradient theory considers the dilatation and stretch of the microstructure in addition to its rotation considered in modified couple stress theory which means that the modified couple stress theory is a special case of the strain gradient theory.

In this paper, a functionally graded strain gradient Euler–Bernoulli beam model is developed. The governing equation of motion and both classical and non-classical boundary conditions are derived using Hamilton principle. Furthermore, a closed-form solution is determined for static deflection of the new model. As case studies, the size-dependent static and free-vibration behavior of a hinged–hinged FG microbeam modeled by the strain gradient theory is assessed.

2. Preliminaries

According to the strain gradient theory proposed by Lam et al. (2003), the stored strain energy U for a linear elastic material occupying region V having infinitesimal deformations is written as (Kong et al., 2009)

$$U = \int_V \bar{u}\,dv = \frac{1}{2}\int_V \left(\sigma_{ij}\varepsilon_{ij} + p_i\gamma_i + \tau^{(1)}_{ijk}\eta^{(1)}_{ijk} + m^s_{ij}\chi^s_{ij}\right)dV, \tag{1}$$

where

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{2}$$

$$\gamma_i = \varepsilon_{mm,i}, \tag{3}$$

$$\eta^{(1)}_{ijk} = \frac{1}{3}(\varepsilon_{jk,i} + \varepsilon_{ki,j} + \varepsilon_{ij,k}) - \frac{1}{15}\delta_{ij}(\varepsilon_{mm,k} + 2\varepsilon_{mk,m}) - \frac{1}{15}[\delta_{jk}(\varepsilon_{mm,i} + 2\varepsilon_{mi,m}) + \delta_{ki}(\varepsilon_{mm,j} + 2\varepsilon_{mj,m})], \tag{4}$$