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Contacts and cracks of complex shapes: Crack-contact dualities and relations between normal and shear compliances

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ABSTRACT

We discuss contacts and cracks of complex shapes, and focus on the following issues:

(1) *Crack-contact duality* – the correspondence between compliances of contacts and cracks of the same shape. For a broad class of shapes (all convex and some concave ones) the correspondence involves shape factor $M \equiv \pi\langle a\rangle\langle a^{-1}\rangle^{-1}/A$ where $a(\phi)$ is the distance from the centroid to boundary points and A is the area. It is controlled mostly by the extent of shape elongation.
(2) *Relations between the normal and shear compliances* of cracks and contacts. The two are relatively close, and this has implications for the anisotropy due to multiple cracks or contacting rough surfaces. It also allows extension of the elasticity-conductivity connections to the shear compliances;
(3) For the *overlapping shapes* with known solutions for each of the component shapes, a simple "summation rule" is suggested.
(4) Comparison of *two approximate methods* of finding compliances of non-elliptical domains – of Fabrikant and of Boyer–Greenwood.

1. Introduction

In applications, contacts and cracks rarely have circular or elliptical shapes for which exact solutions exist. We consider contacts and cracks of complex shapes and focus on their *overall* elastic characteristics (rather than full stress fields), i.e. punch settlement in contact problems, and the average displacement discontinuity in crack problems. These quantities are relevant for the stiffness of contacting rough surfaces, and for effective elastic properties of materials with cracks, respectively.

In this context, we mention two theoretical tools of the general character: (1) the modification theorem of Hill (1963) that, being applied to a crack, bounds its compliance contribution by the ones of the inscribed and circumscribed shapes; it implies, in particular, that shape details such as boundary jaggedness or sharpness of various corner points are of little consequence for crack compliances; (2) the internal variables formalism of Rice (1975) that, being applied to a crack, relates its compliance to stress intensity factors (SIFs) so that available solutions for SIFs can be utilized (Kachanov & Sevostianov, submitted for publication). Both tools can also be applied to contacts, by treating them as external cracks (as explored by Sevostianov & Kachanov (2008)). In the context of cracks, we mention estimates of crack compliances for several specific classes of shapes (Fabrikant, 1989; Grechka & Kachanov, 2006; Mear, Sevostianov, & Kachanov, 2007; Sevostianov & Kachanov, 2002).

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The present work focuses on the following issues.

(1) *Crack-contact duality* – the correspondence between compliances of contacts and cracks having the same shape;
(2) *Relations between the normal and shear compliances* of cracks and contacts. They are of importance for the overall anisotropy due to multiple cracks or contacting rough surfaces. They also allow the extension of the *normal* stiffness-conductivity connection for rough surfaces to the *shear* stiffness;
(3) For *overlapping shapes*, with known solutions for each of the shape components, a simple "summation rule" is suggested;
(4) We compare *two approximate methods* of finding compliances of non-elliptical punches. The method of Fabrikant (1989) yields approximate analytical expressions for punch settlements and crack opening displacements; it works for those shapes where the distance from the centroid to boundary points is a unique function of the polar angle; this covers convex shapes and some concave ones. In the method of Boyer (2001) and Greenwood (1966), a contact of arbitrary shape is broken into a number of microcontacts and their collective electric *resistance* is found by a simple technique. This method was reformulated, in the context of contact *compliances*, by Sevostianov and Kachanov (2009) who also extended it to the *shear* compliances.

2. Background results

We give a brief overview of results on crack- and punch compliances that are relevant for the present work.

2.1. Cracks: general relations and results for the elliptical shape

Crack contributions to the effective elastic properties are determined by displacement discontinuities across their surfaces. Of interest are, therefore, relations between the discontinuities and applied loads.

For any *flat* crack (unit normal $\boldsymbol{n}$ to a crack surface A is constant along A), its contribution to the average, over representative volume V, strain is given by

$$\Delta\boldsymbol{\varepsilon} = \frac{A}{2V}(\boldsymbol{bn} + \boldsymbol{nb}) \tag{2.1}$$

where $\boldsymbol{b} = \langle \boldsymbol{u}^+ - \boldsymbol{u}^- \rangle$ is the vector of average displacement discontinuity. This formula is an immediate consequence of a footnote remark of Hill (1963); in the explicit form, it has been used since early 1970's (see, for example, Salganik (1973)). In the linear elastic formulation (relevant to sufficiently small loads that do not produce either crack propagation or significant crack closures), vector $\boldsymbol{b}$ is a linear function of applied stress and hence can be written as

$$\boldsymbol{b} = \boldsymbol{n}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{B} \tag{2.2}$$

where $\boldsymbol{\sigma}$ is a constant "remotely applied" stress (the stress that would exist in volume V in absence of the crack). Here, $\boldsymbol{B}$ is symmetric second-rank tensor that can be called *the displacement discontinuity tensor* (Kachanov, 1992). The compliance contribution tensor of a flat crack – the fourth-rank tensor $\boldsymbol{H}$ that gives the extra strain due to the crack (2.1) in terms of applied loads, $\Delta\varepsilon_{ij} = H_{ijkl}\sigma_{kl}$ – has the form $\boldsymbol{H} = (1/V)\boldsymbol{nBn}$ (appropriately symmetrized). In other words, the extra compliance due to presence of the crack $\Delta S_{ijkl} = H_{ijkl}$.

Tensor $\boldsymbol{B}$ has three principal directions of crack compliance; if vector $\boldsymbol{n}\cdot\boldsymbol{\sigma}$ is parallel to one of them, the resulting $\boldsymbol{b}$-vector has the same direction as $\boldsymbol{n}\cdot\boldsymbol{\sigma}$. If the material is isotropic – as is assumed hereafter – (or, more generally, orthotropic with the crack parallel to one of the planes of elastic symmetry) then one of the principal directions is normal to the crack so that

$$\boldsymbol{B} = B_N\boldsymbol{nn} + B_s\boldsymbol{ss} + B_t\boldsymbol{tt} \tag{2.3}$$

where mutually orthogonal unit vectors $\boldsymbol{s}$ and $\boldsymbol{t}$ lie in the crack plane. The normal, B_N, and the shear, B_s and B_t, crack compliances give the average displacement discontinuities in the directions produced by uniform loads applied in the same directions. The average, over tangential directions, shear compliance

$$B_T \equiv \langle B_\tau \rangle = (B_s + B_t)/2 \tag{2.4}$$

is yet another quantity of importance, in the context of multiple cracks of random shapes.

If the crack shape has an axis of geometrical symmetry (a pear-like shape, for example), this axis and the normal to it constitute the principal axes of $\boldsymbol{B}$. For the shapes that are tangentially isotropic with respect to the elastic response (besides circles, this covers shapes having symmetries of all equilateral polygons, including squares) $\boldsymbol{B}$ has the form

$$\boldsymbol{B} = B_N\boldsymbol{nn} + B_T(\boldsymbol{I} - \boldsymbol{nn}) \tag{2.5}$$