International Journal of Engineering Science 51 (2012) 74–89

Contents lists available at SciVerse ScienceDirect

International Journal of Engineering Science

journal homepage: www.elsevier.com/locate/ijengsci

Steady homogeneous turbulence in the presence of an average velocity gradient

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ARTICLE INFO

Article history:
Received 19 March 2011
Received in revised form 14 October 2011
Accepted 20 October 2011
Available online 17 November 2011

Keywords:
Lyapunov analysis
von Kármán–Howarth equation
Velocity difference statistics
Anomalous scaling

ABSTRACT

We study the homogeneous turbulence in the presence of a constant average velocity gradient in an infinite fluid domain, with a novel finite-scale Lyapunov analysis, presented in a previous work dealing with the homogeneous isotropic turbulence.

Here, the energy spectrum is studied introducing the spherical averaged pair correlation function, whereas the anisotropy caused by the velocity gradient is analyzed using the equation of the two points velocity distribution function which is determined through the Liouville theorem. As a result, we obtain the evolution equation of this velocity correlation function which is shown to be valid also when the fluid motion is referred with respect to a rotating reference frame. This equation tends to the classical von Kármán–Howarth equation when the average velocity gradient vanishes.

We show that, the steady energy spectrum, instead of following the Kolmogorov law $\kappa^{-5/3}$, varies as κ^{-2}. Accordingly, the structure function of the longitudinal velocity difference $\langle \Delta u_r^n \rangle \approx r^{\zeta_n}$ exhibits the anomalous scaling $\zeta_n \approx n/2$, and the integral scales of the correlation function are much smaller than those of the isotropic turbulence.

1. Introduction

Although the Kolmogorov law $E(\kappa) \approx k^{-5/3}$ represents the main result of the isotropic turbulence, there are many experimental evidences and theoretical arguments indicating that this is not the only spectrum observed in the fully developed turbulence of incompressible fluids (Baroud, Plapp, She, & Swinney, 2002; Brissaud, Frisch, Leorat, Lessieur, & Mazure, 1973; Gordienko & Moiseev, 2001; Moffat, 1978).

For example, in Brissaud et al. (1973) and Moffat (1978), it is shown through the dimensional analysis, that the energy spectrum in the presence of an average velocity gradient $\partial U/\partial y$ can follow the law $\approx k^{-7/3}$ in the inertial subrange. This is a particular result arising from the assumption that the energy spectrum is linear in $\partial U/\partial y$. More in general, assuming that the energy spectrum is proportional to $(\partial U/\partial y)^\beta$ with $\beta > 0$, the Buckingham theorem states that $E(\kappa) \approx \kappa^{-5/3 - 2/3\beta}$, and different scaling exponent are possible.

Gordienko and Moiseev (2001) studied the forced driving turbulence, where the forcing term can have various origins. The authors remarked that there are two dimensionless parameters, characterizing the forcing term, which influence the shape of the energy spectrum and are responsible for the anomalous spectra. They showed that, in a certain interval of variation of one of these parameters, the spectrum follows the Kolmogorov law, whereas for an opportune choice of it, the spectrum behaves like κ^{-2} in the inertial subrange.

These different scaling are caused by the shear rate which leads to the development of coherent fluid structures. These are streaky structures, due to the stretching of the vortex lines, which exhibit the maximum dimension along the stream

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doi:10.1016/j.ijengsci.2011.10.013

direction (Lee, Kim, & Moin, 1990). In Lee et al. (1990), the authors remark that these streaky structures influence the spanwise correlation of streamwise velocity components, being a non monotonic function which becomes negative at high values of the spanwise distances.

Further, in Baroud et al. (2002), the authors experimentally analyzed the statistics of the longitudinal velocity difference Δu_r in a closed cylindric tank which rotates around its symmetry axis at a given spin rate. The turbulence is generated by pumping the flow in the tank through two concentric rings of 120 holes each, placed at the bottom of the tank, where the source ring is the internal one. This generates an average radial flow that, combined with the spin rate, determines a Coriolis force whose magnitude varies with the distance from the rotation axis. As a consequence, a counter rotating flow and an average velocity gradient with respect to the tank frame is observed (Baroud et al., 2002). The authors found that Δu_r presents the anomalous scaling $\langle \Delta u_r^n \rangle \approx r^{\zeta_n}$ with $\zeta_n \simeq n/2$, in contrast with the Kolmogorov law ($\zeta_n \simeq n/3$), and that $E(\kappa) \approx \kappa^{-2}$.

The present work studies the homogeneous turbulence in an infinite fluid domain in the presence of an average velocity gradient $\nabla_{\mathbf{x}} \mathbf{U}$, using the finite-scale Lyapunov analysis, proposed by de Divitiis (2010), and de Divitiis (2011) for studying the homogeneous isotropic turbulence.

In the first section, we define the spherical part of the velocity correlation tensor R_{ij}, and we derive the evolution equation for R_{ii} from the Navier–Stokes equation with $\nabla_{\mathbf{x}} \mathbf{U} \neq \mathbf{0}$.

To study the effect of $\nabla_{\mathbf{x}} \mathbf{U}$ on the anisotropy and on R_{ij}, the evolution equation for the pair distribution function is derived from the Liouville theorem, assuming that the statistical equilibrium corresponds to the condition of isotropic turbulence when the kinetic energy rate is equal to zero. From this equation, the steady velocity correlation tensor is expressed in function of the average velocity gradient and of the maximal finite-scale Lyapunov exponent and, in particular, the Boussinesq closure for the Reynolds stress is obtained.

Finally, the equation for the spherical averaged longitudinal correlation function is determined, whose solutions depend on the average velocity gradient. Moreover, we show that this equation is still valid when the fluid motion is refereed with respect to a non-inertial rotating frame of reference. The steady solutions of this equation are numerically calculated for different Taylor-Scale Reynolds number and several results are presented. We found that $E(\kappa) \approx \kappa^{-2}$ in the inertial subrange, thus the statistical moments $\langle \Delta u_r^n \rangle \approx r^{\zeta_n}$ exhibit the anomalous scaling $\zeta_n \approx n/2$, whereas the integral scales of the longitudinal correlation function are much lesser than those of the isotropic turbulence. In the case of homogeneous turbulence in the presence of a steady shear rate, the spanwise correlation function of the streamwise velocity component is also calculated.

2. Analysis

This section analyzes the homogeneous turbulence with an uniform average velocity gradient $\nabla_{\mathbf{x}} \mathbf{U}$.

The fluid velocity, measured in the reference frame $\Re$, is $\mathbf{v} = \mathbf{U} + \mathbf{u}$, where $\mathbf{U} \equiv (U_x, U_y, U_z)$ and $\mathbf{u} \equiv (u_x, u_y, u_z)$ are, average and fluctuating velocity, respectively. The velocity correlation tensor is defined as $R_{ij} = \langle u_i u'_j \rangle$, being u_i and u'_j the velocity components of $\mathbf{u}$ calculated at $\mathbf{x}$ and $\mathbf{x}' = \mathbf{x} + \mathbf{r}$, where the brackets denote the average on the statistical ensemble of $\mathbf{u}$ and $\mathbf{u}'$, and $\mathbf{r}$ is the separation distance (Kármán & Howarth, 1938; Batchelor, 1953).

In order to determine the evolution equation of R_{ij}, we start from the Navier–Stokes equations, written for the fluctuating velocity (Batchelor, 1953), in the points $\mathbf{x}$ and $\mathbf{x}'$

$$
\begin{aligned}
\frac{\partial u_i}{\partial t} &= -\frac{\partial u_i u_k}{\partial x_k} - \frac{\partial U_i u_k}{\partial x_k} - \frac{\partial u_i U_k}{\partial x_k} - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \nabla^2 u_i \\
\frac{\partial u'_j}{\partial t} &= -\frac{\partial u'_j u'_k}{\partial x'_k} - \frac{\partial U'_j u'_k}{\partial x'_k} - \frac{\partial u'_j U'_k}{\partial x'_k} - \frac{1}{\rho}\frac{\partial p'}{\partial x'_j} + \nu \nabla'^2 u'_j
\end{aligned}
\tag{1}
$$

where p is the fluctuating pressure and $\mathbf{U}'$ is

$$
\mathbf{U}' = \mathbf{U} + \nabla_{\mathbf{x}} \mathbf{U}\ \mathbf{r} \tag{2}
$$

being $\mathbf{U}$ and $\nabla_{\mathbf{x}} \mathbf{U}$ assigned quantities. The repeated index indicates the summation with respect the same index. The evolution equation of R_{ij} is determined by multiplying first and second equation by u'_j and u_i, respectively, summing the so obtained equations, and calculating the average on the statistical ensemble (Batchelor, 1953):

$$
\frac{\partial R_{ij}}{\partial t} = T_{ij} + P_{ij} + 2\nu \nabla^2 R_{ij} - \frac{\partial U_i}{\partial x_k} R_{kj} - \frac{\partial U_j}{\partial x_k} R_{ik} + \frac{\partial R_{ij}}{\partial r_k}(U_k - U'_k) \tag{3}
$$

being

$$
T_{ij}(\mathbf{r}) = \frac{\partial}{\partial r_k}\left\langle u_i u'_j (u_k - u'_k)\right\rangle, \quad P_{ij}(\mathbf{r}) = \frac{1}{\rho}\left(\frac{\partial \langle p u'_j \rangle}{\partial r_i} - \frac{\partial \langle p' u_i \rangle}{\partial r_j}\right) \tag{4}
$$

where $\partial\langle \cdots \rangle / \partial x_i \equiv -\partial \langle \cdots \rangle / \partial r_i$ and $\partial \langle \cdots \rangle / \partial x'_i \equiv \partial \langle \cdots \rangle / \partial r_i$. Making the trace of Eq. (3), we obtain the following scalar equation