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# Micropolar continuum mechanics of fractal media

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ABSTRACT

This paper builds on the recently begun extension of continuum thermomechanics to fractal media which are specified by a fractional mass scaling law of the resolution length scale $R$. The focus is on *pre-fractal media* (i.e., those with lower and upper cut-offs) through a technique based on a dimensional regularization, in which the fractal dimension $D$ is also the order of fractional integrals employed to state global balance laws. In effect, the governing equations are cast in forms involving conventional (integer-order) integrals, while the local forms are expressed through partial differential equations with derivatives of integer order but containing coefficients involving $D$ and $R$, as well as a surface fractal dimension $d$. While the original formulation was based on a Riesz measure—and thus more suited to isotropic media—the new model is based on a product measure capable of describing local material anisotropy. This measure allows one to grasp the anisotropy of fractal dimensions on a mesoscale and the ensuing lack of symmetry of the Cauchy stress. This naturally leads to micropolar continuum mechanics of fractal media. Thereafter, the reciprocity, uniqueness and variational theorems are established.



## 1. Introduction

Mandelbrot's seminal work on fractals (Mandelbrot, 1982) was first followed, outside mathematics itself, by the condensed matter physicists who focused on the effects of fractal geometries on bulk material responses (Feder, 1988). That work concentrated on explaining physical phenomena and properties of materials whose fractal (non-Euclidean) geometry plays a key role, but a field theory—as an analogue of continuum physics/mechanics—has been sorely lacking. Some progress in that respect has recently been made by mathematicians (Kigami, 2001; Strichartz, 2006) looking at classical problems, like Laplace's or heat equation, on fractal (albeit self-similar and non-random) sets. Also, various specialized models have also been developed for particular problems like wave scattering at fractals (Berry, 1979), fracture mechanics (Carpinteri, Chiaia, & Cornetti, 2004), or geomechanics (Dyskin, 2004).

A new step in the direction of continuum physics and mechanics, relying on dimensional regularization, was taken by Tarasov (Tarasov, 2005a, 2005b, 2005c), who developed continuum-type equations of conservation of mass, momentum and energy for fractal porous media, and on that basis studied several fluid mechanics and wave motion problems. In principle, one can then map a mechanics problem of a fractal (which is described by its mass $D$ and surface $d$ fractal dimensions plus the spatial resolution $R$) onto a problem in the Euclidean space in which this fractal is embedded, while having to deal with coefficients explicitly involving $D$, $d$ and $R$. As it turns out, $D$ is also the order of fractional integrals employed to state global balance laws. This approach's great promise stems from the fact that much of the framework of continuum mechanics/physics may be generalized and partial differential equations (with derivatives of integer order) may still be employed

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(Ostoja-Starzewski, 2007a, 2007b). Prior research has already involved an extension to continuum thermomechanics and fracture mechanics, a generalization of extremum and variational principles, turbulent flows in fractal porous media, elastodynamics under small or finite strains and wavefronts in viscoelastic materials (Demmie & Ostoja-Starzewski, in press; Joumaa & Ostoja-Starzewski, in press; Ostoja-Starzewski, 2008, 2009a, 2009b; Ostoja-Starzewski, & Li, 2009).

Whereas the original formulation of Tarasov was based on the Riesz measure—and thus more suited to isotropic media—the model proposed here is based on a *product measure* introduced very recently (Li & Ostoja-Starzewski, 2009). That measure grasps the anisotropy of fractal geometry (i.e., different fractal dimensions in different directions) on mesoscale, which, in turn, leads to asymmetry of the Cauchy stress. This leads to a framework of micropolar mechanics of fractal materials, formulated here for the case of small strains and rotations, building on the model first presented in Li and Ostoja-Starzewski (2010).

## 2. Product measures and basic integral theorems

We begin with a fractal material whose mass $m$ obeys a power law

$$m(R) = kR^D, \quad D < 3. \tag{2.1}$$

Here $R$ is the length scale of measurement (or resolution), $D$ is the fractal dimension of mass, and $k$ is a proportionally constant. Certainly, (2.1) can be applied to a *pre-fractal*, i.e., a fractal-type, physical object with lower and upper length scale cutoffs. Proceeding in the vein of dimensional regularization, Tarasov (Tarasov, 2005a, 2005b, 2005c) used a *fractional integral* to represent mass in a 3D region

$$m(W) = \int_W \rho(\mathbf{r})dV_D = \int_W \rho(\mathbf{r})c_3(D,\mathbf{r})dV_3, \tag{2.2}$$

where the first and second equalities, respectively, involve fractional (Riesz-type) integrals and conventional integrals, while the coefficient $c_3(D,\mathbf{r}) = |\mathbf{r}|^{D-3}2^{3-D}\Gamma(3/2)/\Gamma(D/2)$ provides a transformation between the two. The Riesz fractional integral (or Riesz potential) defines an inverse for a fractional power of the Laplace operator (Samko, Kilbas, & Marichev, 1993). The numeric factor in $c_3(D,\mathbf{r})$ has been scaled to ensure that in the limit $D \to (3-0)$, $c_3(D,\mathbf{r}) \to 1$.

In order to deal with generally anisotropic rather than isotropic media, as done in Ostoja-Starzewski (2009b), Ostoja-Starzewski and Li (2009), Li and Ostoja-Starzewski (2009), we replace (2.1) by a more general power law relation with respect to each coordinate

$$m(x_1,x_2,x_3) \sim x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3}, \tag{2.3}$$

whereby the mass distribution is specified via a product measure

$$M(x_1,x_2,x_3) = \iiint \rho(x_1,x_2,x_3)d\mu(x_1)d\mu(x_2)d\mu(x_3), \tag{2.4}$$

Here the length measurement in each coordinate is provided by

$$d\mu(x_k) = c_1^{(k)}(\alpha_k,x_k)dx_k, \quad k = 1,2,3. \tag{2.5}$$

Then, the total fractal dimension $D$ of mass $m$ is $\alpha_1 + \alpha_2 + \alpha_3$, while a volume coefficient replacing $c_3(D,r)$ of (2.2) is

$$c_3 = c_1^{(1)}c_1^{(2)}c_1^{(3)} = \prod_{i=1}^{3} c_1^{(i)}. \tag{2.6}$$

With a cubic element characterized by the coefficient $c_3(D,r)$ we associate three surface coefficients, each $c_2$ specified by the normal vector along the respective axis; Fig. 1. Thus, $c_2^{(k)}$ associated with the surface element $S_d^{(k)}$ is

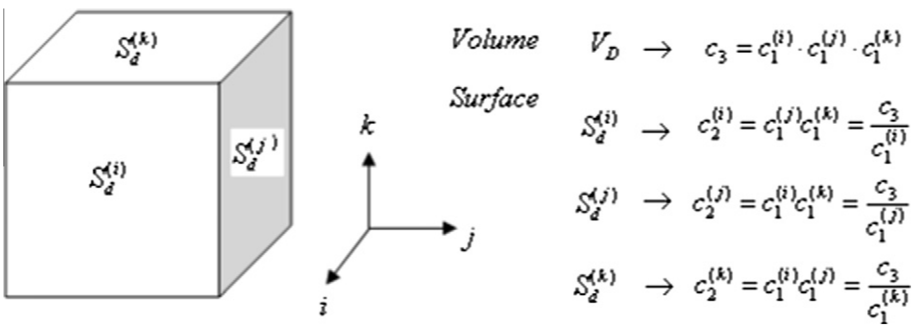


**Fig. 1.** Constructing the coefficients $c_2^{(k)}$ and $c_3$ via the product measure.