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A note on Eringen's moment balances

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ABSTRACT

The present study provides a comparison of Eringen's [Eringen, A.C. (1970). Balance laws of micromorphic mechanics. *International Journal of Engineering Science, 8*, 819–828] general moment balances for micromorphic continua with Germain's [Germain, P. (1973). The method of virtual power in continuum mechanics. Part 2: Microstructure. *SIAM Journal on Applied Mathematics, 25*, 556–575] momentum balances based on virtual work principles, and with those derived in the present paper by a two-scale Fourier analysis of heterogeneous media. It has not been possible to establish a clear-cut correspondence between Eringen's balances and either of the latter, partly because Eringen's balances involve a mixture of surface and volume averages over microdomains.

There is disagreement between the last two methods, arising from the fact that Germain's treatment involves spatial gradients not occurring in the elementary two-scale Fourier analysis. A brief discussion is given of the possible extension of the latter to achieve agreement with the former.

As a separate matter, a construction of path-moments of density fields serves to establish a source-flux duality in continuum balances, which *inter alia* establishes a fairly direct connection between Newton's and Cauchy's laws and provides an expression for stress suggested by the statistical mechanics of point-particles.

1. Introduction

Among his extensive and widely recognized works on continuum mechanics, A.C. Eringen made numerous important contributions to the foundations and application of the theory of *micromorphic continua*.

We recall that, as an extension of the celebrated Cosserat theory, a micromorphic continuum is one whose material particles are endowed with additional degrees of freedom beyond those enjoyed by classical continua. In the case of mechanics, this includes general *microdeformation*[1] superimposed on the translational motion of the *simple* continuum (Truesdell & Noll, 1965).

In a long-running series of publications, well summarized by himself and others (Eringen, 1970, 1992; Germain, 1973), Eringen develops general balance equations for the associated micromorphic continuum. His technique consists basically of the fragmentation of a simple continuum into disjoint sub-bodies or "microdomains", in which variations in field quantities occur on scales much shorter than that of the overlying micromorphic fields.

One merit of Eringen's approach is the concrete physical interpretation it lends to certain fields as moments of familiar mechanical quantities. We take this a step further in the opening paragraphs of the following section, by recalling the equivalent moments for discrete point-particle systems (Goddard, 1998). As a further merit, Eringen's basic technique anticipates,

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[1] Although Eringen often specializes to homogeneous deformation, he recognizes that the balances which are the subject of the present paper allow for arbitrary microdeformation.

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in many respects, the "two-scale" modeling employed by numerous workers for the homogenization of heterogeneous continua (Allaire, 1992; Kevorkian & Cole, 1996; Mielke & Timofte, 2008).

On the other hand, Eringen's results involve field variables defined by integrals over the surface of microdomains, which would seem to imply a possibly unwanted dependence on microdomain geometry. Furthermore, as pointed out by Germain (1973), it is not clear how Eringen's generalized momentum balances are related to balances obtained by other means, such as Germain's application of the principle of virtual work,[2] or other treatments of statistical mechanics and micromechanics. For example, based on the latter, Goddard (1998, 2008) has proposed a hierarchy of momentum balances involving pairwise interaction of moment stresses or "hyperstresses". As discussed below, Germain's analysis allows for a somewhat more general kinematics and conjugate hyperstress.

The present paper proceeds from a cursory review of background theory, along with certain results of Eringen (1970) and Germain (1973), to an alternative approach based on a two-scale analysis of Fourier representations and their spatial moments. As evident from several past works on the homogenization of heterogeneous media (Allaire & Conca, 1998; Kunin, 1982; Murdoch & Bedeaux, 2001), Fourier methods provide an obvious and natural tool for multiscale analysis. It is hoped that the present synthesis of methods may serve to provide, in slightly simpler notation, an enhanced appreciation of the various methods of homogenization.

As a more original effort, the final section below explores the duality between generation and flux in continuum balance equations, with results that may *inter alia* serve to diminish, if not banish, what the author views as a questionable dichotomy between Newton's and Cauchy's laws.

2. Background – Continuum balances

In this section we review the basic equations of balance and certain past work on these. As a word on notation, lowercase bold Greek and Fraktur fonts are employed for general tensors, with restricted use of lowercase bold Roman for vectors and uppercase bold Roman for second-rank tensors. The symbol $\hat{=}$ indicates equivalence between a tensor and its components relative to a general basis, designated by Latin indices. For the most part, we display contravariant components relative to a general curvilinear coordinates. Lowercase Greek indices are used to designate particulate entities and other extensive quantities and a colon is indicated to indicate the contraction or scalar product of tensors of rank greater than one. As a prelude to the discussion of continuum balances, we briefly review discrete-particle systems.

2.1. Moments in discrete-particle systems

To lend insight into certain continuum balances and their Fourier analysis to follow, we recall a related work (Goddard, 1998)[3] and consider a system of N point-particles having positions $\mathbf{x}_\alpha$ and linear momenta $\mathbf{p}_\alpha$, $\alpha = 1, \ldots, N$, each satisfying Newton's law $\dot{\mathbf{p}}_\alpha = \mathbf{f}_\alpha$, where $\mathbf{f}_\alpha$ is the sum of external plus interparticle forces. One may readily derive the Galilean-invariant moment balances (Goddard, 1998)

$$\dot{\mathfrak{p}}^{(m)} = \mathfrak{f}^{(m)}, \text{ with } \mathfrak{p}^{(m)} = \sum_\alpha \mathbf{p}'_\alpha \otimes \mathbf{x}'^{\,m}_\alpha, \ \mathfrak{f}^{(m)} = \sum_\alpha \left(\mathbf{f}'_\alpha \otimes \mathbf{x}'^{\,m}_\alpha + \mathbf{p}'_\alpha \otimes \overline{\mathbf{x}'^{\,m}_\alpha}^{\,\cdot} \right),$$

where, here as below,

$$\mathbf{z}^{m+1} = \mathbf{z}^m \otimes \mathbf{z} \hat{=} [z^{i_1} z^{i_2} \ldots z^{i_{m+1}}], \quad m = 0, 1, 2, \ldots, \mathbf{z}^0 := 1, \tag{1}$$

and

$$\mathbf{z}'_\alpha = \mathbf{z}_\alpha - \bar{\mathbf{z}}, \bar{\mathbf{z}} = \sum_\alpha w_\alpha \mathbf{z}_\alpha, \sum_\alpha w_\alpha = 1.$$

The w_α denote a set of constant scalar weights, usually assumed to be $m_\alpha / \sum_\beta m_\beta$, where m_α denotes particle mass. However, the above formulae remain valid under the alternative condition $\bar{\mathbf{z}} = \mathbf{0}, \ \mathbf{z}'_\alpha = \mathbf{z}_\alpha, \ \alpha = \mathbf{1}, \ldots, \mathbf{N}$, yielding non-Galilean-invariant forms.

In any event, the tensors $\mathfrak{f}$ serve as generation or source of the moments $\mathfrak{p}$ and, when these are converted to densities, the above multiparticle system serves as representative of a single "particle" in a multipolar or micromorphic continuum. This is made more evident by the methods of Irving and Kirkwood (1950), Kunin (1982, 1984), where operator density or *distribution* for a particle-specific physical quantity Q and its continuous Fourier transform are given, respectively, by

$$\varrho_Q(t, \mathbf{x}) = \sum_\alpha Q_\alpha \delta(\mathbf{x} - \mathbf{x}_\alpha\{t\}), \quad \text{and } \hat{\varrho}_Q(t, \mathbf{k}) = \sum_\alpha Q_\alpha e^{-\mathbf{k} \cdot \mathbf{x}_\alpha} \equiv e^{-\mathbf{k} \cdot \bar{\mathbf{x}}} \check{\varrho}, \quad \text{with } \check{\varrho} = \sum_\alpha Q_\alpha e^{-\mathbf{k} \cdot (\mathbf{x}_\alpha - \bar{\mathbf{x}})}, \tag{2}$$

[2] In a related treatment of micromorphic electromagnetism (Eringen, 2006), one encounters the same surface integrals, but I have not made the effort to compare with the analysis of Maugin (1980) based on virtual work.

[3] Which presents a slightly different version, in which, incidentally, $\sum_i \dot{\mathbf{A}}_i \delta(\bar{\mathbf{x}} - \mathbf{x})$ in Eq. (26) should read $\dot{\hat{\mathbf{A}}} \sum_i m_i \delta(\mathbf{x}_i - \mathbf{x})$, $\sum_i \mathbf{x}_i$ in Eq. (27) should read $\sum_i m_i \mathbf{x}_i$, $\mathbf{A}$ corresponds to $\boldsymbol{\alpha}$ of the present work, and the roles of Greek and Latin indices are reversed.