International Journal of Engineering Science 49 (2011) 1256–1267

Contents lists available at ScienceDirect

International Journal of Engineering Science

journal homepage: www.elsevier.com/locate/ijengsci

A nonlinear strain gradient beam formulation

M.H. Kahrobaiyan [a], M. Asghari [a,*], M. Rahaeifard [a], M.T. Ahmadian [a,b]

[a] Department of Mechanical Engineering, Sharif University of Technology, Tehran, Iran
[b] Center of Excellence in Design, Robotics and Automation, Sharif University of Technology, Tehran, Iran

ARTICLE INFO

Article history:
Received 30 November 2010
Received in revised form 15 January 2011
Accepted 23 January 2011
Available online 18 February 2011

Keywords:
Strain gradient theory
Nonlinear behavior
Non-classic continuum theory
Mid-plane stretching

ABSTRACT

In this paper, a nonlinear size-dependent Euler–Bernoulli beam model is developed based on a strain gradient theory, capable of capturing the size effect. Considering the mid-plane stretching as the source of the nonlinearity in the beam behavior, the governing nonlinear partial differential equation of motion and the corresponding classical and non-classical boundary conditions are determined using the variational method. As an example, the free-vibration response of hinged-hinged microbeams is derived analytically using the Method of Multiple Scales. Also, the nonlinear size-dependent static bending of hinged-hinged beams is evaluated numerically. The results of the new model are compared with the results based on the linear strain gradient theory, linear and nonlinear modified couple stress theory, and also the linear and non-linear classical models, noting that the couple stress and the classical theories are indeed special cases of the strain gradient theory.

1. Introduction

Nowadays, beams are broadly used in micro- and nano-electromechanical systems (MEMS and NEMS) such as vibration shock sensors (Lun, Zhang, Gao, & Jia, 2006; McMahan & Castleman, 2004), electro-statically excited micro-actuators (Batra, Porfiri, & Spinello, 2008; Hu, Chang, & Huang, 2004; Moghimi-Zand & Ahmadian, 2009; Mojahedi, Zand, & Ahmadian, 2010), micro-switches (Coutu, Kladitis, Starman, & Reid, 2004; Hua et al., 2007), and atomic force microscopes (AFM) (Chang, Lee, & Chen, 2008; Lee & Chang, 2008; Mahdavi, Farshidianfar, Tahani, Mahdavi, & Dalir, 2008; Turner & Wiehn, 2001). Beams used in MEMS, NEMS have the thickness in the order of microns and sub-microns, emphasizing that the size-dependent deformation and vibration behaviors in micro-scale elements have experimentally been observed (Fleck, Muller, Ashby, & Hutchinson, 1994; Lam, Yang, Chong, Wang, & Tong, 2003; Ma & Clarke, 1995; McFarland & Colton, 2005; Stolken & Evans, 1998). On the other hand, the classical continuum mechanic is not capable of prediction and explanation of the size-dependent behaviors which occur in micron- and sub-micron-scale structures. While, non-classical continuum theories such as higher-order gradient theories and the couple stress theory can interpret the size-dependencies.

In 1960s some researchers such as Mindlin, Touplin and Koiter introduced the couple stress elasticity theory as a non-classic theory capable to predict the size effects with appearance of two higher-order material constants in the corresponding constitutive equations. In this theory, beside the classical stress components acting on elements of materials, the couple stress components, as higher-order stresses, are also available which tend to rotate the elements (Koiter, 1964; Mindlin & Tiersten, 1962; Toupin, 1962). Utilizing the couple stress theory, some researchers investigated the size effects in some problems (e.g. Anthoine, 2000). Employing the equilibrium equation of moments of couples beside the classical equilibrium equations of forces and moments of forces, a modified couple stress theory introduced by Yang, Chong, Lam, and Tong (2002), with one higher-order material constant in the constitutive equations. This newly established theory has been employed by some researchers in order to predict the size-dependent static and vibration behaviors of microbeams (see:

* Corresponding author. Tel.: +98 21 66165523; fax: +98 21 66000021.
E-mail address: asghari@sharif.edu (M. Asghari).

0020-7225/$ - see front matter © 2011 Elsevier Ltd. All rights reserved.
doi:10.1016/j.ijengsci.2011.01.006

Asghari, Rahaeifard, Kahrobaiyan, & Ahmadian, 2011; Fu & Zhang, 2010; Kahrobaiyan, Rahaeifard, Asghari, & Ahmadian, 2010; Kong, Zhou, Nie, & Wang, 2008; Ma, Gao, & Reddy, 2008; Park & Gao, 2006; Wang, 2010).

Considering the first and the second derivatives of the strain tensor effective on the strain energy density, Mindlin (1965) introduced a higher-order gradient theory for elastic materials. Fleck and Hutchinson (1993, 1997, 2001) utilized the Mindlin proposition by only considering the first derivative of the strain tensor and called the theory as the strain gradient theory, with the presence of five higher-order material constants in the constitutive equations. In comparison with the couple stress theory, the strain gradient theory contains some additional higher-order stress components beside the classical and couple stresses. Indeed, the couple stress theory is a special case of the strain gradient theory. In other words, by neglecting the additional higher-order stress components; the strain gradient theory is degenerated into the couple stress theory.

In a similar way considered by Yang et al. (2002) for the modification of the couple stress theory, Lam et al. (2003) introduced a modified strain gradient theory with three higher-order material constants in the constitutive equations. This new theory is reduced in a special case to the modified couple stress theory. Moreover in the literature, there is a simplified version of the strain gradient theory of Mindlin (1965) with one higher-order material constant (Altan & Aifantis, 1997), and also a theory called as the strain gradient theory with surface energy (Vardoulakis & Sulem, 1995). Some beam and plate formulations based on these two theories have already been published (for example Lazopoulos, 2004; Papargyri-Beskou, Tsepoura, Polyzos, & Beskos, 2003). It is noted that the strain gradient with surface energy is essentially a different theory from the strain gradient theory of Mindlin (1965). Hereafter, when the strain gradient theory is used in the text, it denotes the theory presented by Lam et al. (2003). Based on the strain gradient theory, the static and dynamic behaviors of Euler–Bernoulli microbeams have been analyzed by Kong, Zhou, Nie, and Wang (2009). They assessed the influence of the ratio of the beam thickness to the material length scale parameter on the static deflection and also natural frequencies of microbeams. Also, the strain gradient based linear Timoshenko beam formulation has been developed by Wang, Zhao, and Zhou (2010).

For many microbeams used in MEMS and NEMS with two immovable supports, the nonlinear behavior is observed due to the mid-plane stretching. The nonlinearity causes the static and vibration results to be changed significantly (Abdel-Rahman, Younis, & Nayfeh, 2002; Choi & Lovell, 1997; Hassanpour, Esmailzadeh, Cleghorn, & Mills, 2010; Mojahedi et al., 2010). Hence, the need for nonlinear analyses seems to be important in these cases. Recently, size-dependent nonlinear Euler–Bernoulli and Timoshenko beams modeled on the basis of the modified couple stress theory have been developed by Xia, Wang, and Yin (2010) Asghari, Kahrobaiyan, and Ahmadian (2010), respectively.

In this paper, employing the strain gradient theory, a nonlinear size-dependent microbeam formulation is developed. In the other words, the aim is the establishing of the nonlinear large deflection strain gradient beam formulation as the next step in a sequential published papers including the linear couple stress (Kong et al., 2008; Park & Gao, 2006), the linear strain gradient (Kong et al., 2009), and the nonlinear couple stress (Xia et al., 2010) beam formulations. Utilizing the derived equations, the nonlinear size-dependent static and free-vibration behaviors of hinged-hinged micro-beams are assessed. The results of the derived nonlinear strain gradient formulation are compared with the answer of the available formulations including the linear strain gradient theory, linear and nonlinear modified couple stress theory, and also linear and nonlinear classical theory, noting that all of these formulations are indeed special cases of the presented nonlinear strain gradient beam formulation.

2. Preliminaries

According to the strain gradient theory proposed by Lam et al. (2003), the stored strain energy U_m in a continuum made of a linear elastic material occupying region Ω with infinitesimal deformations is written as

$$U_m = \frac{1}{2}\int_\Omega (\sigma_{ij}\varepsilon_{ij} + p_i\gamma_i + \tau^{(1)}_{ijk}\eta^{(1)}_{ijk} + m^s_{ij}\chi^s_{ij})dv, \tag{1}$$

where

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{2}$$

$$\gamma_i = \varepsilon_{mm,i}, \tag{3}$$

$$\eta^{(1)}_{ijk} = \frac{1}{3}\left(\varepsilon_{jk,i} + \varepsilon_{ki,j} + \varepsilon_{ij,k}\right) - \frac{1}{15}\delta_{ij}(\varepsilon_{mm,k} + 2\varepsilon_{mk,m}) - \frac{1}{15}[\delta_{jk}(\varepsilon_{mm,i} + 2\varepsilon_{mi,m}) + \delta_{ki}(\varepsilon_{mm,j} + 2\varepsilon_{mj,m})], \tag{4}$$

$$\chi^s_{ij} = \frac{1}{2}(\theta_{i,j} + \theta_{j,i}), \tag{5}$$

$$\theta_i = \frac{1}{2}(curl(\mathbf{u}))_i, \tag{6}$$

In the above equations, u_i, γ_i and θ_i represent the components of the displacement vector $\mathbf{u}$, the dilatation gradient vector $\boldsymbol{\gamma}$, and the infinitesimal rotation vector $\boldsymbol{\theta}$. Also, the components of the strain tensor $\boldsymbol{\varepsilon}$, the deviatoric stretch gradient tensor $\boldsymbol{\eta}^{(1)}$, and the symmetric part of the rotation gradient tensor $\boldsymbol{\chi}^s$ are denoted by ε_{ij}, $\eta^{(1)}_{ijk}$ and χ^s_{ij}. Consider the strain energy density $\bar{u}_m$ as a function of independent kinematic parameters, and let ψ be a kinematic parameter effective on $\bar{u}_m$. Also, denote the parameter which is obtained from $\partial\bar{u}_m/\partial\psi$ by ϕ. We consider ψ and ϕ as dual to each other. The parameters which are