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Path-independent integrals around two circular holes in an infinite plate under biaxial loading conditions

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ABSTRACT

An analytic solution is presented for stresses induced in an infinite plate with two unequal circular holes by remote uniform loadings and arbitrary internal pressures in the holes. The solution is obtained by using the general expression for a biharmonic function in bipolar coordinates. The Airy stress function is decomposed in the sum of a fundamental stress function for an infinite plate remotely loaded, which gives non vanishing tractions on the circular boundaries, and an auxiliary stress function required to satisfy the boundary conditions on the pressures at the edges of the holes, which produces vanishing stresses at infinity. Correspondingly, the variations of the stress concentration factor are determined in terms of the holes geometry and loading conditions. The path independent J_k-($k = 1, 2$), M- and L-integrals are analytically calculated on a closed contour encircling the two holes, under remote loading, in order to evaluate the energy release rates accompanying unit translation, self similar expansion and rotation of the holes, respectively. Results are then presented for varying loading orientation angle, biaxial loading ratio and holes geometry.

1. Introduction

Holes in structural components create stress concentrations and cracks often initiate and propagate from the locations of stress concentration. Therefore, an accurate evaluation of the stress concentration factor (SCF) at the edge of holes is a prerequisite to assure the structural integrity of many structural components and underground constructions, such as perforated plates, boreholes, buried pipes and tunnels. The two-dimensional problem of the stress distribution in an infinite elastic medium containing two circular holes under plane stress or plane strain conditions has attracted the interest of many researchers, who approached the problem by using various methods. Initially, Jeffery (1921) found the most general form of a biharmonic stress function in bipolar coordinates and separated out the terms which give rise to many-valued displacement. He also examined some of the simpler application of the theory, such as a cylinder with eccentric bore and a half-plane with a circular hole. However, he did not solve the problem of an infinite plate with two unequal circular holes because of difficulty in determination of unknown coefficients. Later, a similar approach was used by Ling (1948) to solve the problem of an infinite plate containing two equal circular holes under general far-field stresses. The approach was extended by Dhir (1968) to the case of two equal holes reinforced by a thin elastic ring with no bending stiffness, perfectly bonded around the openings. A comprehensive treatise on the problem of stress concentration around holes was compiled by Savin (1961). By using the Jeffery solution, Radi and Strozzi (2009) recently solved the problem of a circular disk containing a sliding eccentric circular inclusion.

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The case of two unequal circular holes has received minor attention in the past. Haddon (1967) used a conformal mapping and complex variable techniques to obtain a closed form solution for an infinite plate with two unpressurized holes, loaded by a uniaxial tension acting in an arbitrary direction. Iwaki and Miyao (1980) found a closed form solution for stresses induced in an infinite plate with two unequal circular holes by a uniform tension applied at infinity and by internal pressure or uniform shear applied at the boundary of one hole, whereas the other hole is unloaded. They used the basic stress functions for an infinite plate with one circular hole under the applied loads and added an auxiliary stress function which has a singular point at the center of this hole and satisfies the boundary condition of no traction at the edge of the same hole. By following the approach proposed by Green (1940) and based on coordinate transformations between the polar coordinate systems attached to each hole, Hoang and Abousleiman (2008) presented a closed-form solution for the stress distribution in an infinite plate containing two equal or unequal circular holes subjected to uniform stresses at infinity and arbitrary pressures inside the holes. Their approach can be generalized to an arbitrary group of circular holes of any size, although it could result very complicate. The solution obtained by Hoang and Abousleiman (2008) has been recently used to estimate the conditions for wellbore stability during drilling and oil production, taking into account the complexity of the 3D anisotropic stress state and holes geometry (Hoang, Abousleiman, & Al-Tahini, 2010). Except for the works of Iwaki and Miyao (1980) and Hoang and Abousleiman (2008) most of the existing exact solutions apply only to stress-free condition at the boundary of the holes, even if the situation of pressurized holes is more frequent in engineering applications.

The elastic interaction between a main hole and a smaller one is the basic principle of the defense hole method, used to reduce stress concentrations in loaded plates. According to this method, smaller holes are introduced on either side of the original hole to reduce the tensile principal stress near the original hole. By using two-dimensional photoelasticity, Erickson and Riley (1978) determined the optimum size and location of defense holes for a number of plates with different size and orientation of the holes. Later, Meguid (1986), Meguid and Gong (1993) and Meguid and Shen (1992) analytically investigated the interaction between a main hole and arbitrarily located defense hole systems, under uniaxial tension. Their analysis was based upon the Muskhelishvili complex potentials, a superposition procedure and the Laurent series expansion. The latter authors also performed comprehensive numerical and photoelastic investigations and concluded that the introduction of defense holes on either side of the original hole allows smoothing the peak hoop (tangential) stress at the original hole. They pointed out the importance of choosing the exact size and location of the defense holes and found that the largest hoop stress at the main hole occurs when the central line of the two holes is inclined at 30° with respect to the direction of the applied load. By using bipolar coordinates, Davanas (1992) also provided an analytical solution to the problem of the elastic interaction between two pressurized holes with equal sizes as well as in the limit case in which the size of one hole becomes very large. He showed that the elastic interaction is always repulsive, regardless the sign of the surface tractions on the holes. Tsukrov and Kachanov (1997) examined the elastic interactions between holes of various shapes and eccentricities. They noticed that the interaction effects may be maximal not in the ideally symmetric arrangements but in the configurations where the symmetry is slightly perturbed. A short survey of studies of the elastic interaction of two holes in a stretched plate is presented by Panasyuk and Savruk (2009). Recently, Hu and Shen (2008), Hui and Chen (2010) and Zhang and Shen (2011) extended the investigation by considering the effects of surface energy on the interaction between two equal or unequal holes.

Path-independent integrals derived from Noether's theorem in linear elasticity play an important role in the description of multiple defects interaction and damage evolution in brittle materials (Budiansky & Rice, 1973; Knowles & Sternberg, 1972). Physically, the J_k- ($k = 1, 2$), M- and L-integrals can be interpreted as the energy release rate for uniform movements, expansion, and rotation of the defects, respectively. Honein, Honein, and Herrmann (2000) evaluated such integrals for two circular elastic inclusions in anti-plane shear deformation. They showed that two holes attract each other under remote uniform shear loading and the J_k- and M-integrals becomes unbounded as the two holes become infinitely close. However, they noticed that under linearly varying remote shear stress the material force may be attractive or repulsive depending on the distance of separation. Hu and Chen (2009) proposed the use of the M-integral as an effective parameter in describing the configurational change and the evolution of damage of two nearby holes up to coalescence. By considering different orientations of the two holes with respect to the direction of the remote uniaxial tensile loading and comparing the calculated values of the M-integral to the reduction of the effective elastic moduli, Chen (2001, 2002) and Hu and Chen (2009) concluded that the larger the M-integral is, the larger the reduction is. Moreover, they proved that under a monotonically increasing loading, the M-integral always increases with the damage evolution and it is inherently related to the change of the total potential energy for a brittle material, regardless of the detailed damage features. However, the latter authors recognized that the coalescence of two holes can not be governed by a single parameter, e.g. the critical value of the M-integral as proposed by Chang and Peng (2004), because the critical value depends on the relative locations and orientations of the two holes, although it is independent of the detailed coalescence path connecting the two holes. Then, in order to describe complete failure mechanism of a brittle material with many defects, Hu and Chen (2011) proposed the combination of outside variable features, like the M- and L-integrals, with the classical inner variable features defined, e.g., by the effective elastic moduli theory. These works indicate that a bridge between the invariant integrals and the damage mechanics might be established.

Many of the studies on the interactions between two equal or unequal holes have been conducted by numerical methods (Hu & Chen, 2009, 2011; Meguid, 1986; Meguid & Gong, 1993). However, closed-form solutions are preferred for the preliminary design of circular openings, required in many problems in mechanical, aerospace, underground and geotechnical engineering. Exact solutions for stresses and displacements are also required for the validation of numerical methods.