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Successive linear approximation for boundary value problems of nonlinear elasticity in relative-descriptional formulation

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ABSTRACT

A method of successive linear approximation in current configuration for solving boundary value problems of large deformation in finite elasticity is proposed. Instead of using Lagrangian or Eulerian formulation, we can also formulated the problem relative to the current configuration, and linearize the constitutive function at the present state so that it leads to a linear boundary value problem for an incremental time step. Therefore, as linearization at present state proceed in time, problem for large deformation can be formulated. The idea is similar to the Euler's method for differential equations. As an example for the proposed method, numerical simulation of bending a rectangular block into a circular section for Mooney–Rivlin material is given for comparison with the exact solution, which is one of the well-known universal solutions in finite elasticity.

1. Introduction

The constitutive equation of a solid body is usually expressed relative to a preferred reference configuration which exhibits specific material symmetries such as isotropy. The constitutive functions are generally nonlinear. Therefore, for large deformations, it leads to a system of nonlinear partial different equations for the solution of boundary value problems. The problems are usually formulated in referential (Lagrangian) or spatial (Eulerian) coordinates, and numerical solution of nonlinear systems of algebraic equations by Newton's type methods. Alternatively, we shall formulate boundary value problems in the coordinates relative to the configuration at the present time. In other words, we shall take the current configuration as the reference configuration instead of a fixed reference configuration in the Lagrangian formulation. Note that this is not an Eulerian formulation either. The Eulerian coordinates are fixed coordinates in space. We shall call this a relative-descriptional formulation or co-motional formulation, since the problem is described from an observer as if he were attached to the body in its motion.

Deformations relative to the current configuration are called relative deformations which has been considered in standard textbooks of Continuum Mechanics (see Liu, 2002; Truesdell & Noll, 2004), and will be briefly discussed in the formulation of the equation of motion in current coordinates in this paper.

The advantage of using the relative-descriptional formulation is that we can consider a small time step from the present state, so that the constitutive functions can be linearized relative to the present state, and the equation of motion becomes linear in relative displacement. When the present state proceeds in time, a nonlinear finite deformation can be treated as a sequence of linear deformations in the same manner as the usual Euler's method for solving differential equations, i.e., successively at each state, the tangent is calculated and used to extrapolate the neighboring state. This is especially useful and interesting in numerical implementation, since a problem with large deformations can be treated successively as a

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sequence of linear problems for small deformations just as in the theory of linear elasticity, except that the elasticity tensor is not constant, since, at each time step, the constitutive equation has to be updated at the current state of deformation.

There are some well-known exact solutions in nonlinear elasticity for incompressible elastic bodies. One of such solutions, the bending of a rectangular block into a circular sector, is presented for comparison with the results in numerical simulation to illustrate the performance of the proposed method.

For incompressible elastic bodies, the indeterminate pressure must be treated as an independent variable in addition to the displacement vector variable. In order to avoid adding the pressure as an additional variable together with the condition of incompressibility, we shall treat incompressible body as a compressible one by a near-incompressibility assumption which greatly simplify the boundary value problems in numerical computations.

2. Relative description of deformation

Let κ_0 be a reference configuration of an elastic body $\mathcal{B}$, $\mathcal{B}_0 = \kappa_0(\mathcal{B})$ and

$$\boldsymbol{x} = \chi(X, t), \quad X \in \mathcal{B}_0$$

be the motion of the body. Let κ_t be the deformed configuration at time t, $\mathcal{B}_t = \kappa_t(\mathcal{B})$, and

$$F(X,t) = \nabla_X \chi(X,t)$$

be the deformation gradient with respect to the configuration κ_0. In our discussions, we shall always refer to time t as the present time.

Let κ_τ be the deformed configuration at time τ. The deformation $\boldsymbol{\xi} = \chi(X,\tau)$ from κ_0 can be described in the current configuration κ_t at the present time t by

$$\xi = \chi(X,\tau) \ := \ \chi_t(\boldsymbol{x},\tau) \quad \text{for} \quad \boldsymbol{x} = \chi(X,t),$$

where $\chi_t(\cdot,\tau) : \mathcal{B}_t \to \mathcal{B}_\tau$ is called relative deformation. Note that we have used the subscript t to denote the description relative to the present time t following the notation used in Liu (2002) and Truesdell and Noll (2004).

The relative displacement vector $\boldsymbol{u}$ from the current configuration κ_t is given by

$$\boldsymbol{u} = \boldsymbol{\xi} - \boldsymbol{x} = \boldsymbol{u}_t(\boldsymbol{x},\tau) = \chi_t(\boldsymbol{x},\tau) - \boldsymbol{x}, \quad \boldsymbol{x} \in \mathcal{B}_t. \tag{1}$$

Taking the gradients relative to X and $\boldsymbol{x}$ respectively, we obtain

$$H_t(\boldsymbol{x},\tau)F(X,t) = F(X,\tau) - F(X,t), \quad H_t(\boldsymbol{x},\tau) = F_t(\boldsymbol{x},\tau) - I, \tag{2}$$

where I is the identity tensor, and

$$H_t(\boldsymbol{x},\tau) = \nabla_{\boldsymbol{x}} \boldsymbol{u}_t(\boldsymbol{x},\tau), \quad F_t(\boldsymbol{x},\tau) = \nabla_{\boldsymbol{x}} \chi_t(\boldsymbol{x},\tau)$$

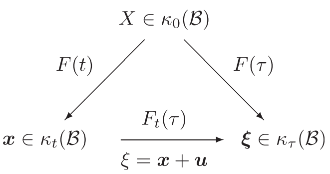
are called the relative displacement gradient and the relative deformation gradient respectively. Note that from (1) and (2), we have

$$H_t(\boldsymbol{x},t) = 0, \quad F_t(\boldsymbol{x},t) = I,$$

and

$$F_t(\boldsymbol{x},\tau) = F(X,\tau)F(X,t)^{-1}, \quad F(X,\tau) = (I + H_t(\boldsymbol{x},\tau))F(X,t). \tag{3}$$

We can represent the situation in the following diagram:



where and hereafter, for simplicity, sometimes only the time dependence is indicated. Position dependence is usually self-evident and will be indicated for clarity if necessary.

Remark on descriptions of motion

Recall that in *Lagrangian description*, a function $f(X,t)$ defined on the motion of a body is defined on the domain $\mathcal{B}_0 \times \mathbb{R}$, in the fixed reference configuration; while in *Eulerian description*, it is defined on the position x occupied by the body, $\hat{f}(\boldsymbol{x},t)$. Note that the domain of the Eulerian description is not $\mathcal{B}_t \times \mathbb{R}$, but rather $\mathcal{B}_t \times \{t\}$.

On the other hand, we have defined the function on $\mathcal{B}_t \times \mathbb{R}$, at time τ relative to the current configuration, as if the function $f_t(\boldsymbol{x},\tau)$ is viewed at the instant τ from an observer attached to the body in its motion at the present time t. This will be called the *relative description* or perhaps as *co-motional description*.