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Distributed-order fractional wave equation on a finite domain. Stress relaxation in a rod

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A B S T R A C T

We study waves in a viscoelastic rod of finite length. Viscoelastic material is described by a constitutive equation of fractional distributed-order type with the special choice of weight functions. Prescribing boundary conditions on displacement, we obtain displacement and stress in a stress relaxation test. We use the Laplace transformation method in the time domain as a tool for solving system of differential and integro-differential equations, that describe the motion of the rod.

1. Introduction

Fractional derivatives have been used in describing physical phenomena such as viscoelasticity, diffusion and wave phenomena. For the detailed account of applications of fractional calculus in viscoelasticity (see Mainardi, 2010; Rossikhin, 2010; Rossikhin & Shitikova, 2010). There are two approaches in formulating differential equations with fractional derivatives in physics and mechanics. In the first approach classical "integer order" differential equations of a process are modified by introducing fractional derivatives instead of integer order ones (see Kilbas, Srivastava, & Trujillo, 2006; Mainardi, 1997; Podlubny, 1999). In the second approach one uses variational principles such as the Hamilton principle as a starting point for deriving equations of a process, where a modification of the classical case is achieved by replacing some (or all) integer order derivatives in Lagrangian density by fractional derivatives of certain kind. Then the resulting Euler–Lagrange equations are equations of a process and they contain both left and right fractional derivatives (see Agraval, 2002; Atanackovic, konjik, & Pilipovic, 2008; Atanackovic & Stankovic, 2007).

In this paper we generalize classical wave equation for one-dimensional elastic body by following the first approach. Recall the classical setting. Consider the equation of motion

$$\frac{\partial}{\partial x}\sigma(x,t)=\rho\frac{\partial^2}{\partial t^2}u(x,t),\quad x\in[0,L],\ t>0, \tag{1}$$

where ρ, σ and u denote density, stress and displacement of a material at a point positioned at x and at a time t, respectively. It is coupled with the Hooke Law

$$\sigma(x,t)=E\varepsilon(x,t),\quad x\in[0,L],\ t>0, \tag{2}$$

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where E is a modulus of elasticity and ε is a strain measure, defined by

$$\varepsilon(x,t) = \frac{\partial}{\partial x}u(x,t), \quad x \in [0,L],\ t > 0. \tag{3}$$

Combining (1)–(3), the classical wave equation is obtained as

$$\frac{\partial^2}{\partial x^2}u(x,t) = \frac{\rho}{E}\frac{\partial^2}{\partial t^2}u(x,t), \quad x \in [0,L],\ t > 0.$$

We propose the generalization of a constitutive equation (2) by replacing it with a constitutive equation which corresponds to a generalized viscoelastic body:

$$\int_0^1 \phi_1(\alpha)\,{}_0D_t^\alpha \sigma(x,t)\mathrm{d}\alpha = E\int_0^1 \phi_2(\alpha)\,{}_0D_t^\alpha \varepsilon(x,t)\mathrm{d}\alpha, \quad x \in [0,L],\ t > 0, \tag{4}$$

where E is a generalized Young modulus (positive constant having dimension of stress), ϕ_1 and ϕ_2 are given functions or distributions and ${}_0D_t^\alpha y$ is the left Riemann–Liouville fractional derivative of a function $y \in AC([0,T])$, for every $T > 0$, of the order $\alpha \in [0,1)$, defined as

$${}_0D_t^\alpha y(t) := \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{y(\tau)}{(t-\tau)^\alpha}\mathrm{d}\tau, \quad t > 0,$$

where Γ is the Euler gamma function. Recall, $AC([0,T])$ denotes the space of absolutely continuous functions (for a detailed account on fractional calculus, see Samko, Kilbas, and Marichev (1993)).

In the case when ϕ_1 and ϕ_2 are distributions, we assume that ϕ_1 and ϕ_2 are compactly supported by $[0,1](\phi_1,\phi_2 \in \mathcal{E}'(\mathbb{R}), \mathrm{supp}\phi_1, \mathrm{supp}\phi_2 \subset [0,1])$. In this case integrals in (4) are as

$$\left\langle \int_{\mathrm{supp}\phi} \phi(\alpha)\,{}_0D_t^\alpha h(t)\mathrm{d}\alpha, \varphi(t) \right\rangle := \left\langle \phi(\alpha), \left\langle {}_0D_t^\alpha h(t), \varphi(t)\right\rangle\right\rangle, \quad \varphi \in \mathcal{S}(\mathbb{R}).$$

For details, see Atanackovic, Oparnica, and Pilipovic (2009). Recall, $\mathcal{S}'_+(\mathbb{R})$ denotes the space of tempered distributions supported by $[0,\infty)$ and $\langle h(t), \varphi(t)\rangle$ denotes the action of a distribution $h \in \mathcal{S}'_+(\mathbb{R})$ on a test function $\varphi \in \mathcal{S}(\mathbb{R})$ (see Vladimirov, 1984).

In (4), ϕ_1 and ϕ_2 denote constitutive functions or distributions that are determined experimentally (see Rogers, 1983; Schiessel, Friedrich, & Blumen, 2000). The constitutive equations of type (4) were used earlier in Atanackovic (2002b), Atanackovic, Pilipovic, and Zorica (2009a), Atanackovic, Pilipovic, and Zorica (2009b), Hartley and Lorenzo (2003). There are number of forms that ϕ_1 and ϕ_2 can take (see e.g. Hartley & Lorenzo, 2003). In the sequel we assume that

$$\phi_1(\alpha) := a^\alpha, \quad \phi_2(\alpha) := b^\alpha, \qquad \alpha \in (0,1),\ a \leqslant b. \tag{5}$$

The restriction $a \leqslant b$ follows from the Second Law of Thermodynamics (see Atanackovic, 2002a, 2003). If $a = b$, then (4) reduces to the Hooke Law. The choice of ϕ_1 and ϕ_2 in the form (5) is the simplest choice guaranteeing dimensional homogeneity. Note that with $\phi_1(\mu) := \delta(\mu) + \tau_\varepsilon^\alpha \delta(\mu - \alpha)$ and $\phi_2(\mu) := E_\infty \tau_\varepsilon^\beta \delta(\mu - \beta)$ (δ denotes the Dirac distribution) we obtain

$$\sigma + \tau_\varepsilon^\alpha\,{}_0D_t^\alpha \sigma = E_\infty \tau_\varepsilon^\beta\,{}_0D_t^\beta \varepsilon, \tag{6}$$

while with $\phi_1(\mu) := \delta(\mu) + \tau_\varepsilon^\alpha \delta(\mu - \alpha)$ and $\phi_2(\mu) := E_0\left(\delta(\mu) + \tau_\sigma^\alpha \delta(\mu-\alpha) + \tau_\sigma^\beta \delta(\mu - \beta)\right)$ we obtain

$$\sigma + \tau_\varepsilon^\alpha\,{}_0D_t^\alpha \sigma = E_0\left(1 + \tau_\sigma^\alpha\,{}_0D_t^\alpha + \tau_\sigma^\beta\,{}_0D_t^\beta\right)\varepsilon. \tag{7}$$

Recall, system (1), (3) and (6), respectively, system (1), (3) and (7), was treated in Rossikhin and Shitikova (2001), respectively, in Rossikhin and Shitikova (2001). Similarly, the constitutive equations used in Rossikhin and Shitikova (2004) could be obtained from (4) by choosing ϕ_1 and ϕ_2 to be the linear combination of the Dirac delta distributions. Also note that the distributed order dissipation of type (4) was also used in the context of one degree of freedom mechanical systems in Atanackovic, Budincevic, and Pilipovic (2005) and Atanackovic and Pilipovic (2005).

Our aim is to find functions u and σ, locally integrable on $\mathbb{R}$ and equal to zero for $t < 0$, so that these functions satisfy (1), (3) and (4), for $x \in [0,L]$ and $t > 0$, as well as the appropriate initial and boundary conditions. For this, we will introduce dimensionless quantities and transform the system (1), (3) and (4), subject to (8) and (9), into the system (10), subject to (11) and (12).

The paper is organized as follows. In Section 2 we introduce dimensionless quantities, proceed by formal calculation and by the use of the Laplace transformation we obtain solutions to (1), (3) and (4) in the convolution form. We impose initial conditions as well as boundary conditions to (1), (3) and (4). Boundary conditions describe a rod that is fixed at one of its ends, while the other end is subject to a prescribed displacement Υ (this is the case of stress relaxation if $\Upsilon = \Upsilon_0 H$, with H being the Heaviside function). Section 3 is devoted to the calculation of the inverse Laplace transformation, which leads to the explicit form of a solution. More precisely, we investigate some properties of functions in order to be able to apply the Cauchy residues theorem, which is used to calculate the inverse Laplace transformation. We obtain displacement u and stress σ for the boundary condition $\Upsilon = \Upsilon_0 H$ in Section 3.1.1, as well as for $\Upsilon = \Upsilon_0 H + F$, where F is an appropriate function supported by $[0,\infty)$, in Section 3.1.2. We conclude that solutions are locally integrable functions supported by $[0,\infty)$.