Chaos, Solitons and Fractals 114 (2018) 130–144

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

Analytic solutions for variance swaps with double-mean-reverting volatility

See-Woo Kim, Jeong-Hoon Kim*

Department of Mathematics, Yonsei University, Seoul 03722, Republic of Korea

A R T I C L E I N F O

Article history:
Received 13 January 2018
Revised 27 May 2018
Accepted 21 June 2018

Keywords:
Variance swap
Gatheral model
Heston model
Double mean reversion
Characteristic function

A B S T R A C T

A three factor variance model introduced by Gatheral in 2008, called the double mean reverting (DMR) model, is well-known to reflect the empirical dynamics of the variance and prices of options on both SPX and VIX consistently with the market. One drawback of the DMR model is that calibration may not be easy as no closed form solution for European options exists, not like the Heston model. In this paper, we still use the double mean reverting nature to extend the Heston model and study the pricing of variance swaps given by simple returns in discrete sampling times. The constant mean level of Heston's stochastic volatility is extended to a slowly varying process which is specified in two different ways in terms of the Ornstein-Uhlenbeck (OU) and Cox-Ingersoll-Ross (CIR) processes. So, two types of double mean reversion are considered and the corresponding models are called the double mean reverting Heston-OU model and the double mean reverting Heston-CIR models. We solve Riccati type nonlinear equations and derive closed form exact solutions or closed form approximations of the fair strike prices of the variance swaps depending on the correlation structure of the three factors. We verify the accuracy of our analytic solutions by comparing with values computed by Monte Carlo simulation. The impact of the double mean reverting formulation on the fair strike prices of the variance swaps are also scrutinized in the paper.

1. Introduction

Since Chicago Board Option Exchange (CBOE) volatility index (VIX) was introduced in 1993, volatility began to be considered as an asset. Trading volume of volatility and variance derivatives has increased rapidly in real market and also a variety of volatility derivatives have been produced. With increasing volatility market, volatility and variance swaps, forward contracts on realized volatility and variance, take possession of the volatility market as a tool for hedging the volatility exposure or trading the spread between realized and implied volatility. In theoretical aspect, variance swaps have several advantages over volatility swaps. Above all, variance swaps have simpler structure than volatility swaps because variance is a square of standard deviation. The payoff of variance swaps increases as volatility rises, which makes beneficial when the volatility is high [1]. Also, it is worth while to note that Demeterfi et al. [2] and Carr and Madan [3] have shown that continuous sampling variance swaps can be replicated by a static portfolio of European options with an equity.

To trade variance swaps, we need the fair strike price calculated under no-arbitrage condition in exchange for realized variance. Heston model [4] that contains the volatility process following the CIR process [5] is used in many studies to derive price of various financial derivatives because of its mathematical advantage [6–9]. In the field of pricing volatility based derivatives, one can find many studies attempting to derive the strike price of variance swaps under the Heston model. For example, Zhu and Lian [10] obtain an analytic formula for variance swaps under the Heston model by using a partial differential equation and Fourier transform method. Swishchuk [11] prices variance and volatility swaps under the Heston model by using a probabilistic approach. Zheng and Kwok [12] derive a closed form solution of generalized variance swaps under the Heston model. Cao et al. [13] extend the study of variance swaps to the case of the Heston model with stochastic interest rates. Also, there are studies of variance swaps under other stochastic volatility models. For example, Bernard and Cui [14] price variance swaps under Hull-White [15] and Schobel-Zhu [16] models apart from the Heston model using asymptotic method. Little and Pant [17] use the finite difference method to study variance swaps under a general stochastic volatility model.

The studies of variance swaps quoted above are based on one-factor stochastic volatility model. In the context of option pricing, it is well-known that one factor stochastic volatility model has a major drawback when it comes to fitting an arbitrage-free implied volatility surface to market data, especially at short time to ma-

* Corresponding author.
E-mail address: jhkim96@yonsei.ac.kr (J.-H. Kim).

https://doi.org/10.1016/j.chaos.2018.06.024

turities. To overcome this inadequacy, it is usually required to add additional parameters, which allows the model to be more flexible. This can be accomplished by allowing the parameters to be time dependent or enriching the volatility process. Christoffersen et al. [18] specify a two-factor structure for the volatility and suggest the so-called double Heston model. Also, we note that Pun et al. [19] adopt a multi-factor stochastic volatility model with jumps to price variance swaps. Swishchuk [20] studies variance swaps under assorted one-factor and multi-factor models with delay. There is a three factor model of Gatheral [21], called the double mean reverting (DMR) model or the Gatheral model, as an interesting extension of the Heston model. This model has a strong advantage that it can be successfully calibrated to both VIX options and SPX options simultaneously. One drawback of this model is, however, that no analytic solution for variance swaps and European options exists and thus calibration may be slow.

To obtain an analytic solution for variance swaps under a three factor stochastic volatility model, we incorporate the double mean reverting nature into the Heston model and specify the DMR model to a certain degree using both OU [22] and CIR processes for the stochastic mean level of Heston's volatility. We call the corresponding double mean reverting models the double mean reverting Heston-OU model and the double mean reverting Heston-CIR model, respectively. We first obtain the characteristic function of each model and then derive analytic solutions for the fair strike prices of variance swaps. The analytic solutions are given by exact or approximate ones depending on the correlation structure of the three factors. Accuracy of our analytic solutions is tested using Monte Carlo simulation. The impact of our type of double mean reverting formulation on the variance swap price is given with numerical experiment.

The rest of the paper is structured as follows. In Section 2, we introduce double mean reverting dynamic structures of an underlying asset. In Sections 3 and 4, we obtain (exact and approximate) analytic solutions for the fair strike price of a variance swap under two different specified models by using generalized characteristic function. In Section 5, we perform a numerical experiment to test validity of our solutions using Euler scheme Monte Carlo (MC) simulation and investigate the price sensitivity with respect to the model parameters. In Appendix, we show technical calculations including how to solve some ordinary differential equations (ODEs) and Taylor approximations for expectation of some processes.

2. Underlying dynamics

The choice of variance dynamics is crucial in handling volatility derivatives. We extend the Heston model by replacing the constant mean level of stochastic variance with a stochastic process. The new model is a specific form of the DMR model.

2.1. The DMR model

Using the underlying price $S(t)$ and Brownian motions $W_i(t)$ $(i = 1, 2, 3)$, the DMR model is given by

$$\begin{aligned} dS(t) &= \sqrt{\nu(t)}S(t)dW_1(t),\\ d\nu(t) &= \kappa(\theta(t) - \nu(t))dt + \sigma\nu(t)^{\gamma_1}dW_2(t),\\ d\theta(t) &= \alpha(\beta - \theta(t))dt + \eta\theta(t)^{\gamma_2}dW_3(t), \end{aligned}$$

where κ, σ, α, β, η, γ_1 and γ_2 are constants. This form of dynamics is beneficial in terms of matching with market data but arduous to derive analytic solution for financial derivatives. For the purpose of obtaining an analytic solution for the fair strike price of variance swaps, we specify the framework of the DMR model while still extending the one-factor Heston model in the following way.

$$\begin{aligned} dS(t) &= \mu S(t)dt + \sqrt{\nu(t)}S(t)dW_1(t),\\ d\nu(t) &= \kappa(\theta_0 + \theta(t) - \nu(t))dt + \sigma\sqrt{\nu(t)}dW_2(t),\\ d\theta(t) &= \alpha(\beta - \theta(t))dt + \eta(t,\theta(t))dW_3(t), \end{aligned} \tag{2.1}$$

with $dW_1(t)dW_2(t) = \rho_1 dt$, $dW_1(t)dW_3(t) = \rho_2 dt$ and $dW_2(t)dW_3(t) = \rho_3 dt$ under a market probability measure P, where ρ_i is a constant with $-1 \le \rho_i \le 1$ for $i = 1, 2, 3$ and θ_0 denotes the minimum long-term mean level of variance.

According to the Girsanov theorem [23], there exists a risk-neutral probability measure Q equivalent to the market probability measure P under which (2.1) is transformed into

$$\begin{aligned} dS(t) &= rS(t)dt + \sqrt{\nu(t)}S(t)d\hat{W}_1(t),\\ d\nu(t) &= (a_0 + a_1\theta(t) + a_2\nu(t))dt + \sigma\sqrt{\nu(t)}d\hat{W}_2(t),\\ d\theta(t) &= (b_0 + b_1\theta(t))dt + \eta(t,\theta(t))d\hat{W}_3(t), \end{aligned} \tag{2.2}$$

where $a_0 = \kappa\theta_0$, $a_1 = \kappa$, $a_2 = -\kappa - \lambda_1$, $b_0 = \alpha\beta$ and $b_1 = -\alpha - \lambda_2$ and $\hat{W}_i(t)$ $(i = 1, 2, 3)$ are Brownian motions. Here, each λ_i $(i = 1, 2)$ denotes the premium of volatility risk as named in Heston's paper [4].

To obtain a dynamic system with mutually independent Brownian motions, we apply the Cholesky decomposition [24] to (2.2). Then system (2.2) is expressed as

$$\begin{bmatrix} dS(t)/S(t)\\ d\nu(t)\\ d\theta(t) \end{bmatrix} = \begin{bmatrix} r\\ a_0 + a_1\theta(t) + a_2\nu(t)\\ b_0 + b_1\theta(t) \end{bmatrix} dt + \Sigma \times C \times \begin{bmatrix} dW_1^*(t)\\ dW_2^*(t)\\ dW_3^*(t) \end{bmatrix}, \tag{2.3}$$

where Σ and C are

$$\Sigma = \begin{bmatrix} \sqrt{\nu(t)} & 0 & 0\\ 0 & \sigma\sqrt{\nu(t)} & 0\\ 0 & 0 & \eta(t,\theta(t)) \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0\\ \rho_1 & \sqrt{1-\rho_1^2} & 0\\ \rho_2 & \frac{\rho_3 - \rho_1\rho_2}{\sqrt{1-\rho_1^2}} & \sqrt{1 - \rho_2^2 - \left(\frac{\rho_3-\rho_1\rho_2}{\sqrt{1-\rho_1^2}}\right)^2} \end{bmatrix}$$

satisfying

$$CC^T = \begin{bmatrix} 1 & \rho_1 & \rho_2\\ \rho_1 & 1 & \rho_3\\ \rho_2 & \rho_3 & 1 \end{bmatrix}$$

and $W_1^*(t)$, $W_2^*(t)$ and $W_3^*(t)$ are mutually independent Brownian motions under the measure Q satisfying

$$\begin{bmatrix} d\hat{W}_1(t)\\ d\hat{W}_2(t)\\ d\hat{W}_3(t) \end{bmatrix} = C \times \begin{bmatrix} dW_1^*(t)\\ dW_2^*(t)\\ dW_3^*(t) \end{bmatrix}.$$

By substituting $x(t) = \log(S(t))$, system (2.3) becomes

$$\begin{bmatrix} dx(t)\\ d\nu(t)\\ d\theta(t) \end{bmatrix} = \begin{bmatrix} r - \nu(t)/2\\ a_0 + a_1\theta(t) + a_2\nu(t)\\ b_0 + b_1\theta(t) \end{bmatrix} dt + \Sigma \times C \times \begin{bmatrix} dW_1^*(t)\\ dW_2^*(t)\\ dW_3^*(t) \end{bmatrix}. \tag{2.4}$$

To obtain an analytic solution for variance swaps, we choose two types of $\eta(t, \theta(t))$ in this paper.