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Scale-free and small-world properties of hollow cube networks☆

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ABSTRACT

In this paper, we construct the evolving networks from hollow cube in fractal geometry by encoding. We set the unit cubes as nodes of network, where two nodes are neighbors if and only if their corresponding cubes have common surface. We also study some characteristics of the network, such as degree distribution, clustering coefficient and average path length. We obtain this network with small world and scale-free properties by the self-similar structure.

1. Introduction

With the rapid development of fractal geometry in the past ten years, the study of which is not confined to the theory of dimension and has been expanded to the complex networks [1–3]. Song et al. [4,5] reveal that many real networks have self-similarity and fractality, including the world wide web, social networks, protein interaction networks and cellular networks, and so on [6–8]. The fractal network is the result of the research combined with fractal science and complex network theory.

On the other hand, we discover some evolution network models having self-similar fractal characteristics [9–11]. The Iterated Function System IFS is a useful technique to understand fractals, which has always been used in constructing the fractal networks. Zhang et al. construct evolving networks modeled from Sierpinski gasket by taking the line segments as nodes [12,13]. Wang et al. construct evolving networks modeled from Sierpinski carpet by taking the unit squares as nodes [14]. Besides Zhang et al. [15]construct the networks produced from Vicsek fractals.

In addition most networks display the scale-free property [16,17], which means that the probability that a randomly selected node with exactly k links decays as a power law, following $P(k) \sim k^{-r}$. What's more, lots of networks have the small-world behavior [18]. In the complex network knowledge, the small world characteristic is that the complex network has not only a short average path length, but also a large average clustering coefficient.

In this paper, we will construct the evolving networks modeled on the hollow cube by encoding from IFS. For our networks, we check their scale-free and small-world effect.

2. Encoding the hollow cube

Consider a solid cube $Q = [0, 1]^3$, each edge of the cube is divided equally into three parts, and the whole cube is divided into the same 27 small cubes. Each cube is $\frac{1}{27}$ of the initial cube Q. Take away the central small cube so that, the remaining 26 small cubes construct a hollow cube. Then, we code small cubes $S_a(Q), S_b(Q), \ldots, S_z(Q)$ with 26 English letters $A = \{a, b, c, \ldots, x, y, z\}$. The next thing is to represent the self similar fractal set by iteration method. The self-similar set $E = \cup_{i\in A} S_i(E)$ with respect to the IFS $S_{i i\in A}$ is called the hollow cube. As shown in Fig. 1. Then the Hausdorff dimension of it is $log_3 26$.

Denote

$$S_\sigma = S_{i_1 i_2 \ldots i_k} = S_{i_1} \circ S_{i_2} \circ \ldots \circ S_{i_k} \quad and \quad Q_\sigma = S_\sigma(Q). \tag{2.1}$$

Where the word $\sigma = i_1 i_2 \ldots i_k$ is composed of the letters in A. Then we denote the length of σ with $|\sigma| = k$, we also call Q_σ a k-stage basic cube with side length 3^{-k}. Through this way, we obtain a natural method to encode the hollow cube. We can encode the solid cube Q_σ by the word σ. We have the following constraints:

(i) For any integer $k \geq 1$, we let $\triangle_k = A^k = \{a, b, \ldots, y, z\}^k$ be the collection of all words of length k.

(ii) For $k = 0$, we have $\triangle_0 = \{\emptyset\}$, where $\emptyset$ is the empty word with $|\emptyset| = 0$, $S_\emptyset(Q) = Q$.

(iii) Denoting $\sigma \prec \tau$, which means σ is a prefix of τ, if $\sigma = i_1 i_2 \ldots i_k$ and $\tau = i_1 i_2 \ldots i_k j_1 j_2 \ldots j_m$.

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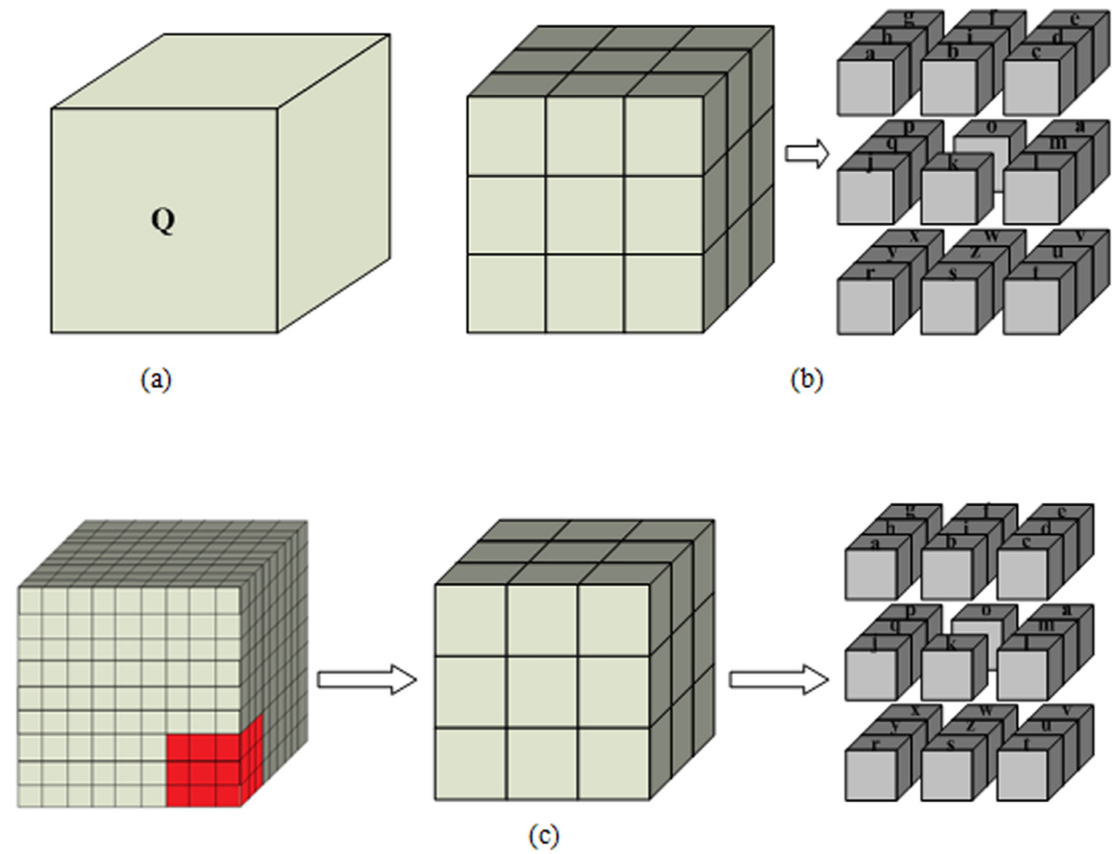

Fig. 1. (a)is the initial solid cube Q, (b)is encoding 1-stage basic cubes, (c)is the 2-step of construction of hollow cube.

(iv) Defining σ is the father of τ, if $\sigma \prec \tau$ and $|\sigma|+1=|\tau|$. For any word $\sigma = i_1 i_2 \dots i_k$, we denote by $\sigma^- = i_1 i_2 \dots i_{k-1}$. Anyone has 26 children while any child has only a unique father in the setting.

3. The construction of evolving networks

Fix t, we will construct a evolving network G_t with node set $V_t = \bigcup_{k=0}^{t} \triangle_k$. Then we have the total number of nodes in the network G_t.

$$\#V_t = 1 + 26 + 26^2 + \ldots + 26^t = \frac{26^{t+1}-1}{25}. \quad (3.1)$$

Given distinct words σ and τ being nodes in the network, there is an edge between σ and τ, denoted $\sigma \sim \tau$, if and only if the intersection of Q_σ and Q_τ is a square. See Fig. 2 for G_2 network.

Remark 3.1. If $\tau = \sigma\beta$, then $\sigma \sim \tau$ if and only if all letters of β appear in one of the following six sets

$\{a, b, c, d, e, f, g, h, i\}, \{r, s, t, u, v, w, x, y, z\}, \{a, g, h, j, p, q, r, x, y\}$
$\{c, d, e, l, m, n, t, u, v\}, \{e, f, g, n, o, p, v, w, x\}, \{a, b, c, j, k, l, r, s, t\}.$

Any word σ has a unique shortest word remarked as $f(\sigma)$ such that $f(\sigma) \prec \sigma$ and $f(\sigma) \sim \sigma$. For a word $\sigma = \tau\beta$, where β is the maximal suffix such that all its letters lie in the one of the above sets. Then we have $f(\sigma) = \tau$. Iterating f, we can obtain a sequence from σ to $\emptyset$.

$$\sigma \sim f(\sigma) \sim f^2(\sigma) \sim \ldots \sim f^n(\sigma) = \emptyset. \quad (3.2)$$

Writing $\omega(\sigma) = n$ respects the shortest steps from σ to $\emptyset$. For example, the word $\sigma = abcfkjnoqxzwxaqp = (abcf)(kj)(no)(q)(xzw)(xaqp)$, so we have $f^6(\sigma) = \emptyset$.

4. Degree distribution

Through the previous description, we can get the following facts.

Claim 4.1. (i) *For any given $k > 0$ and $\sigma \in \triangle_k$, σ has at most 6 and at least 3 neighbors in $\triangle_k$.*

$$3 \le \#\{\tau \in \triangle_k : \sigma \sim \tau\} \le 6.$$

Furthermore, we have

$$\#\{\tau \in \triangle_k : \sigma \sim \tau \text{ and } \tau^- = \sigma^-\} \le 4,$$
$$\#\{\tau \in \triangle_k : \sigma \sim \tau \text{ and } \tau^- \ne \sigma^-\} \le 3.$$

(ii) *For any given $h > k$, the number of neighbors of σ in $\triangle_h$.*

$$8 \times 3^{h-k} + 6 \times 9^{h-k} + 2 \le \#\{\tau \in \triangle_h : \sigma \sim \tau\}$$
$$\le 12 \times 9^{h-k} - 12 \times 3^{h-k} + 8.$$

(iii) *For any given $h < k$, the number of neighbors of σ in $\triangle_h$.*

$$\#\{\tau \in \triangle_h : \sigma \sim \tau\} \le 3.$$

(i), (iii) two points are geometric and intuitive, (ii) it is not difficult to use the inclusion-exclusion principle

We use $deg(\sigma)$ to denote respectively the numbers of neighbors of σ.

Lemma 4.1. *Given $t > 1$ and $k < \frac{t}{2}$, we have*

$$\{\sigma : deg(\sigma) \ge 27 \times 9^{t-k}\} = \{\sigma : |\sigma| < k\}. \quad (4.1)$$

Proof. Suppose $\sigma \in V_i$, using Claim 4.1., then
(1) if $i < k$,

$$deg(\sigma) \ge (8 \times 3^{t-k+1} + 3 \times 9^{t-k+1} + 2)$$