Chaos, Solitons and Fractals 113 (2018) 25–30

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

Automatic generation of bounds for polynomial systems with application to the Lorenz system[☆]

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ARTICLE INFO

Article history:
Received 21 December 2017
Revised 11 April 2018
Accepted 11 May 2018

MSC:
34D45
37C70
37C75

Keywords:
Invariant sets
Positive invariant sets
Lyapunov techniques
Quantifier elimination
Lorenz system

ABSTRACT

This study covers an analytical approach to calculate positive invariant sets of dynamical systems. Using Lyapunov techniques and quantifier elimination methods, an automatic procedure for determining bounds in the state space as an enclosure of attractors is proposed. The available software tools permit an algorithmizable process, which normally requires a good insight into the systems dynamics and experience. As a result we get an estimation of the attractor, whose conservatism only results from the initial choice of the Lyapunov candidate function. The proposed approach is illustrated on the well-known Lorenz system.

1. Introduction

A dynamical system may have an attractor which implies a neighborhood around the attractor on which all trajectories are bounded. Thus, it is interesting to ask if such a neighborhood can be described analytically. From a geometric point of view, this means we intend to find a subset of the state space with some special properties, which is also called finding a compact invariant set or calculating an enclosure of the attractor. A standard procedure for calculating such a bound is to employ positive Lyapunov-like functions. However, there are two major restrictions to employing Lyapunov-like functions. A first is that the algebraic form as well as the parameterization of the function offers a considerable degree of choice, which usually makes finding a suitable candidate a matter of (mostly human) trail-and-error. A second is that calculating symbolic or numerical values of the bound on the Lyapunov-like candidate function that gives an estimate of the compact invariant set commonly requires human insight, experience, and frequently substantial algebraic manipulations. The approach we present here aims at circumventing these problems by proposing an automatic and algorithmizable procedure. This is done using quantifier elimination (QE) methods. The term quantifier elimination covers several methods [1–3] to reformulate quantified formulas into a quantifier free equivalent. This idea has already been applied for stability analysis [4–7], model verification [8] as well as controller design [8–10].

The method we propose calculates an analytic expression of compact positive invariant sets. Moreover, if a dynamical system is dissipative and may consequently possess one (or even several) attractors, then a positive invariant set may contain at least one of them. However, attractors are invariant sets with additional requirements. They have to be compact, they are not dividable into two invariant, disjoint subsets and they need an attractive neighborhood. This consequently applies to chaotic attractors, which have a complicated geometry, and for which compact positive invariant sets can provide an enclosure. Thus, calculating attractor enclosures by compact positive invariant sets is related to, but differs in methodology and objective from computing analytic expressions of attractors themselves, as shown by calculating invariant measures and fractal dimension for the 2D Lorenz map [11,12], or almost-invariant sets and invariant manifolds for the Lorenz system [13]. These differences in objective and methodology stem

☆ Partially supported by the German Academic Scholarship Foundation.
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https://doi.org/10.1016/j.chaos.2018.05.012
0960-0779/

from this paper using Lyapunov–like function for calculating attractor enclosures, while calculating attractor approximations has been shown by either using a geometric description of the dynamics by invariant manifolds or a probabilistic description of dynamics by transfer operators or almost-invariants sets [11–15].

Although our method is generally applicable to dynamical systems with polynomial description, we specifically apply it to the Lorenz system to have a comparison with previous results. The Lorenz equations [16,17] are arguable one of the best-known and most-studied dynamical systems that exhibit chaotic solutions. This also includes several works on ultimate bounds, compacts sets, or attractor enclosures [17–24]. Apart from analyzing a property of the Lorenz system, calculating bounds is also a possible starting point for applications, for instance estimating the fractal dimension [25] or the Hausdorff dimension of the Lorenz attractor [24]. Attractor enclosures have additionally been used for the tracking of periodic solutions, stabilization of equilibrium points and synchronization [24,26,27].

The paper is structured as follows: In Section 2 we introduce our approach with briefly recalling quantifier elimination and calculating bounds on trajectories using Lyapunov-like functions. We also discuss how quantifier elimination can be used to obtain such bounds. The method is applied to the Lorenz system in Section 3. We calculate spherical and elliptical bounds with fixed and variable center points and show that our method can be used to reproduce, algebraically verify and partly improve bounds known from previous works [17,20,21]. In Section 4 we derive some conclusions.

2. Computation of bounds for dynamical systems by quantifier elimination

2.1. Real quantifier elimination

Before we illustrate the proposed method let us briefly introduce some mathematical preliminaries on quantifier elimination (QE), cf. [28,29], starting with a simple example to delineate the main ideas of QE.

Let us consider the quadratic function $g(x) = a_2x^2 + a_1x + a_0$. The question if a parameter constellation $u = (a_0, a_1, a_2)$ exists such that the function values $g(x)$ are always positive can be formulated using the quantified expression

$$\exists\, a_2, a_1, a_0\ \forall x : g(x) > 0,$$

which can easily be answered with `true`. If we are interested in all parameter constellations u, which result in $g(x) > 0$, we utilize QE. Therefore, we omit the quantifiers for u to generate an equivalent expression in these quantifier-free variables

$$\forall x : g(x) > 0.$$

Applying a QE method to the problem we get

$$(a_1 = 0 \vee 4a_2a_0 - a_1^2 \neq 0) \wedge a_0 > 0 \wedge -4a_2a_0 + a_1^2 \leq 0.$$

Thus, we get exact conditions which are equivalent to the previous formula. After presenting the necessary fundamentals of QE, it is next shown how these techniques can be applied to estimate positive invariant sets.

In the following, we introduce the concept of quantifier elimination in a more formal way.

An *atomic formula* is an expression of the form

$$\phi(x_1, \dots, x_k)\ \tau\ 0 \tag{1}$$

with a relation $\tau \in \{>, =\}$, where $\phi \in \mathbb{Q}[x_1, \dots, x_k]$ is a polynomial in the variables $x_1, \dots, x_k$ with rational coefficients. A combination of atomic formulas (1) with the Boolean operators $\wedge$, $\vee$, $\neg$ is called a *quantifier-free formula*. With these standard operators we can express all other Boolean operators such as equivalence ($\Leftrightarrow$) or implication ($\Rightarrow$) and augment the list of relations for (1) to $\{<, \leq, >, \geq, =, \neq\}$.

Let $F(u,v)$ be a quantifier-free formula in the variables $u = (u_1, \dots, u_k)$ and $v = (v_1, \dots, v_l)$. A *prenex formula* is an expression

$$G(u, v) := (Q_1 v_1) \dots (Q_l v_l)\, F(u, v) \tag{2}$$

with quantifiers $Q_i \in \{\exists, \forall\}$ for $i = 1, \dots, l$. The variables v are called *quantified* and the variables u are called *free*, respectively. Thus, the parameters $\{a_2, a_1, a_0\}$ gives the set u and $\{x\}$ gives the set v in the before described example of the quadratic equation. The quantifiers occurring in (2) can be eliminated [30–32]. This process is referred to as *quantifier elimination*. The following theorem is a direct consequence of the well-known Tarski–Seidenberg-Theorem [33, pp. 69–70]:

Theorem 1 (Quantifier elimination over the real closed field)**.** *For every prenex formula $G(u,v)$ there exists an equivalent quantifier-free formula $H(u)$.*

The first algorithm to determine such a quantifier-free equivalent was presented by Tarski itself. Unfortunately, this algorithm was not applicable because its computational complexity can not be bounded by any stack of exponentials. The first procedure which could be applied to non-trivial problems is *cylindrical algebraic decomposition* (CAD) [28]. This algorithm mainly consists of four steps. The first decompose the space in so-called cells in which every polynomial has a constant sign. Secondly, these cells are gradually projected from $\mathbb{R}^n$ to $\mathbb{R}^1$. These projections are cylindrical and algebraic. The conditions of interest are evaluated in $\mathbb{R}^1$ in the third step and the results are finally lifted to $\mathbb{R}^n$. Due to the universal applicability to the input sets of polynomials, this algorithm and its improvements (e.g. [34]) are still often used. Nevertheless, in the worst case the computational effort is doubly exponential in the number of variables [35].

The second commonly used procedure is *virtual substitution* [2,36,37]. The problem $\exists v$: $F(u,\ v)$ is solved with a formula substitution equivalent, where a is substituted with terms of an elimination set. This procedure is just applicable to linear, quadratic and cubic polynomials, but the resulting complexity is "just" exponential in the number of quantified variables. Furthermore, the resulting conditions are often very large and redundant such that a subsequent simplification is necessary.

A third frequently applied method for QE is based on *real root classification* (RRC). The number of real roots in a given interval can be computed using Sturm or Sturm–Habicht sequences. Based on that idea, formulations to eliminated quantifiers can be generated [38–40]. As in the case of virtual substitution the resulting output formulas are often very large and redundant such that a subsequent simplification is need as well. However, very effective algorithms can be achieved, especially for sign definite conditions $\forall v \geq 0 \Rightarrow f(u,\ v) > 0$, see [40].

To carry out the quantifier elimination we used the open-source software packages QEPCAD [34,41], and REDLOG [42]. The later package is part of the computer algebra system REDUCE. For both tools, the resulting quantifier-free formulas can be simplified with the tool SLFQ [43].

The computations were carried out on a standard PC with Intel® Core™ i3-4130 CPU at 3.4 GHz and 32GiB RAM under the Linux system Fedora 25 (64 bit). For QE we used the advanced virtual substitution method from [44] (i.e., function `rlqe` with the switch `on ofsfvs`). The authors made the source code of prototype implementations publically available on Github [45] under the GNU GPL v3.0 in order to allow a verification of the presented results.