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Implicit and fractional-derivative operators in infinite networks of integer-order components

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ABSTRACT

Complex engineering systems may be considered to be composed of a large number of simple components connected to each other in the form of a network. It is shown that, for some network configurations, the equivalent dynamic behavior of the system is governed by an *implicit* integro-differential operator even though the individual components themselves satisfy equations of integer order. The networks considered here are large trees and ladders with potential-driven flows and integer-order components in the branches. It has been known that in special cases the equivalent operator for the overall system in the time domain is a fractional-order derivative. In general, however, the operator is implicit without a known time-domain representation such as a fractional derivative would have, and can only be defined as a solution to an operator equation. These implicit operators, which are a generalization of commonly known fractional-order derivatives, should play an important role in the analysis and modeling of complex systems. This paper illustrates the manner in which they naturally arise in the modeling of integer-order networked systems.

1. Introduction

Dynamic modeling involves an operator in time. Though it must be pointed out that derivatives with respect to spatial variables can be similarly treated, for the rest of the paper we will assume time derivatives. Some physical systems have been known to have behavior governed by fractional-order time derivatives [1–3] in phenomena such as viscoelasticity, mass diffusion, fractal media, and vibrations, among others [4–7]. It is shown here that there are systems in which a broader class of operators, called *implicitly-defined integro-differential*—or simply implicit—operators, appear and which, in special cases, become the usual integer or fractional-order time derivatives that can be explicitly defined. Currently there are many integer- and fractional-order models for engineering systems, but what this paper describes are physical systems in which implicit operators appear naturally.

Specifically, it is shown that the first-principles modeling of large collections of interacting integer-order components naturally leads to an implicit representation of the operator describing the dynamics of the system. These implicit operators appear in the modeling of networked mechanical systems, such as those shown in Figs. 1, 3 and 4. The components of these systems are governed by the laws of mechanics which are usually integro-differential of *integer* order. However, it has sometimes been mistakenly assumed that, because the components are of integer order, it follows that the larger system is also of integer order. It will be shown that that is not the case and that implicit operators, and their special cases the integer and fractional-order derivatives, are basic tools for the modeling of complex systems.

Implicit operators in themselves are not entirely new. For example, classical texts such as [8] and [9] respectively consider convergent sequences of operators and operator functions such as the exponential. Furthermore, matrix equations and inequalities commonly arise in some fields, such as the Riccati equation in control theory where the solution represents terms in the formulation of an optimal control problem [10]. Implicit matrix operators defined by quadratic and other nonlinear equations also find physical applications, and numerical methods are often used for their solution [11–13]. However, these examples do not include integro-differential operators and the modeling of a system from its basic physics as is done here.

The mathematical foundation upon which we develop our results is the operational calculus of Mikusiński [14,15]. His idea is to define the product of two functions by convolution, which results in another function. The inverse operation, “division,” is not necessarily a function however, and is more generally an

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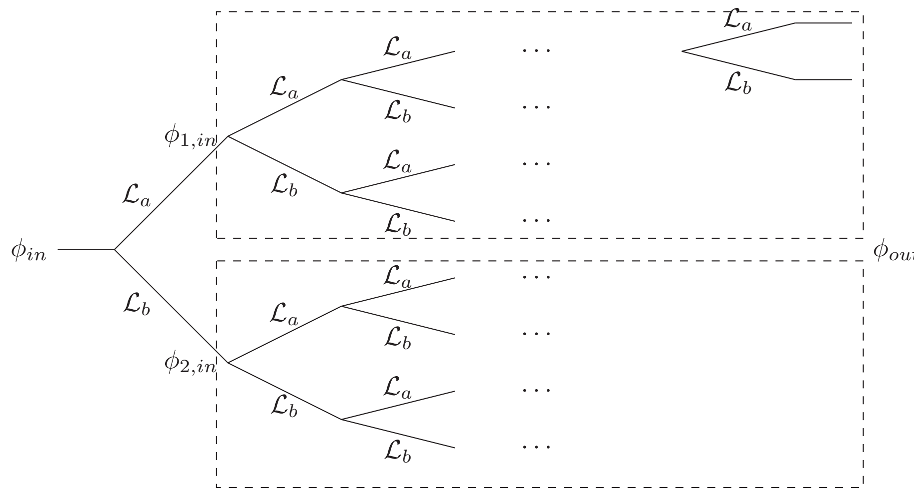

Fig. 1. Infinite tree composed of operators $\mathcal{L}_a$ and $\mathcal{L}_b$. When the number of generations is very large or infinite, then the process occurring between ϕ_{in} and ϕ_{out} will be the same as that between $\phi_{1,\,in}$ and ϕ_{out}, or between $\phi_{2,\,in}$ and ϕ_{out}. In other words, the transfer functions relating the input to the output of the networks in the dashed boxes will be equal to the transfer function for the whole system. A simplified version of this circuit is shown in Fig. 2.

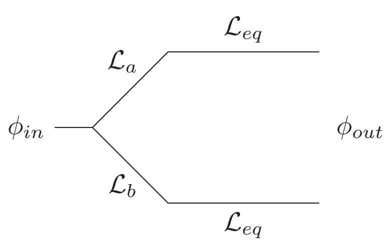

Fig. 2. Components in boxed areas in Fig. 1 replaced by their equivalents.

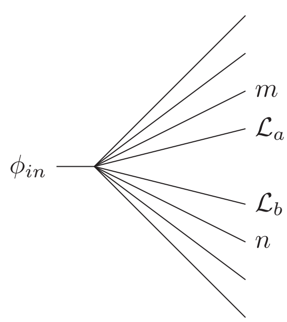

Fig. 3. First generation of multi-furcating tree network. Each junction has m components with $\mathcal{L}_a$ and n components with $\mathcal{L}_b$ connecting to a junction of a subsequent generation.

operator; this is analogous to extending the set of integers to rational numbers. From this development, operations on operators and operator equations follow. In this framework, expressing the dynamics of a system as the solution to an operator equation with operator coefficients is rigorous and makes sense. Specifically, Mikusiński [14,15] develop the framework for *operational functions* and their derivatives (but not, to our knowledge, an implicit function theorem for them). We can develop equations of the form $G(\mathcal{L}) = 0$ from the basic physics of the problem, where $\mathcal{L}$ is an integro-differential time operator and $G(\mathcal{L})$ represents an operational function (scalar variables and functions will be written in Roman italic, constants in Greek, and integro-differential operators in calligraphic) that defines it. We cannot explicitly "solve" for $\mathcal{L}$ and obtain a time-domain differential or integral equation. We will show that expressions of this type describe the dynamics of the kind of systems we are modeling.

One contribution of this paper is to define implicit integro-differential operators in a general setting in the manner of Mikusiński [14,15], and another is to show the manner in which they can naturally arise in the modeling of large-scale networked systems. The latter is done by the explicit development of mechanical examples in which these implicit operators appear. Such large-scale systems are of increasing importance in the engineering and scientific communities, recognized by much research focus on cyber-physical systems and the internet of things, and hence their modeling is becoming similarly important. The use of an operator describing the dynamic relationship between input and output of such a large-scale system that may only be implicitly defined is a new concept that poses significant future challenges.

2. Definitions

Implicit integro-differential operators can be defined in a manner similar to implicit functions and matrix operators [16], and used for the direct mathematical modeling of complex systems. In this way, fractional derivatives become a special case of implicit operators, and mathematical modeling of complex systems can be extended beyond the use of fractional calculus.

2.1. Numbers

There is an analogy between numbers and operators that will be exploited here. An equation such as

$$G(x) = 0, \tag{1}$$

where $G : \mathbb{R} \to \mathbb{R}$ is a known function, has solutions x (x may not be unique if G is nonlinear, as for example if it is a quadratic). More importantly, the integer, rational, irrational or transcendental nature of x is determined by the form of G; a G that is linear in x defines natural, zero, whole, integer and rational numbers, a second- or higher-degree polynomial G permits irrational and complex numbers, and a non-polynomial G a transcendental number. The binary operations of addition (and hence subtraction) and multiplication (and division) of numbers can also be defined.

A number x multiplied by itself several times can be written as x^n, where n is an integer. Numbers satisfy $x^n x^m = x^{n+m}$ for any integer n and m. The rules of manipulation of indices include the following: $x^0 = 1,\ x^1 = x,\ x^{-n} = 1/x^n,\ x^n \times x^m = x^{n+m},\ x^n/x^m = x^{n-m}$, $(x^n)^m = x^{nm}$. These rules can be extended to x^α for any real number α by *assumption* or *postulation*, although clearly α no longer