Pickover biomorphs and non-standard complex numbers

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ABSTRACT

In this study Pickover biomorphs are analysed as being dependent on the chosen complex number system in which iterations of analytic functions are performed. Moran's spatial autocorrelation function and two forms of entropy, the Shannon entropy and the sample entropy, are chosen in order to find correlations and measure complexity in Pickover biomorphs. These turn out to be strongly correlated and low-entropy objects with a fractal dimension between 1.4 and 2. It is shown that there is a strong maximum in correlation and a strong minimum in entropy for the case of Galilean complex numbers corresponding to the square of the generalised imaginary unit being equal to zero.

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1. Introduction

Mandelbrot sets, Julia sets [1–4] and Pickover biomorphs [5–9] belong to a group of mathematical and graphical fractal objects which are of particular fascination to both scientists and artists. Remarkably some Pickover biomorphs morphologically resemble unicellular protozoa with their various organelles or even aquatic crustaceans. Together with better known related objects such as Mandelbrot sets, Pickover biomorphs are an interesting example of very complex structures emerging from the application of simple rules which determine the application of a transformation of the complex plane by iterations of analytic functions. Due to the simplicity of such rules, Pickover biomorphs have attracted some mild attention in the field of theoretical biology [10,11]. These applications in the simulation of evolution set out the most important context of the present study. In one sense, Pickover biomorphs may also be said to be a fascinating by-product of complex dynamics in mathematics [12].

It is not, however, easy to analyse Pickover biomorphs mathematically partly because (to the authors' best knowledge) there is not even a heuristic mathematical definition. Pickover biomorphs are in fact defined by a family of algorithms which create them or even directly as merely graphical objects which resemble certain invertebrates. Thus, this study solely relates to images obtained from more or less standard algorithms. In order to characterise biomorphs quantitatively, two kinds of measures have been used. Firstly, as a measure of spatial correlations, the Moran autocorrelation function has been chosen. Secondly, as a measure of complexity, two versions of entropy, the Shannon entropy and the sample entropy have been applied. The second type of entropy is considered to be an approximation of the Kolmogorov–Sinai entropy. It turns out that even though the autocorrelation function and the entropy measure different characteristics of the biomorphs they still seem to be complementary to each other.

There are two obvious considerations at this point, namely, (a) whether it is of any use to extend the notion of biomorphs to non-standard systems of complex numbers, and (b) what exactly the purpose of application of the machinery of statistics and complexity-related measures to them is. For (a), we believe that from the point of view of investigations in evolution theory exemplified by Mojica et al. [10] it is useful to have at one's disposal structures with sharply defined qualitative changes of properties which can be obtained just by changing one very well defined parameter, in our case here the square of the imaginary unit. For (b), when simulating the evolution of simple forms with the help of genetic algorithms as in [10] it is important to know whether

the "evolution" leads to greater or smaller complexity, stronger or weaker morphological correlations, and even lower or higher fractal dimensions.

2. Non-standard systems of complex numbers

It is well known that one of the fundamentally important procedures in mathematics is the pairing of two real numbers to form a complex number. Multiplication in the set of complex numbers can be defined using Hamilton's construction or alternatively, any complex number z can be written as $z = x + iy$ with $i^2 = -1$. One can then formally operate with complex numbers as with real ones under the assumption of distributiveness of multiplication with respect to addition and substituting in for i^2. Let us note that if i is understood as just a symbol (rather than a number) with a defined square, the construction of complex numbers can easily be extended to cases where i^2 is different to -1. In fact there exist three canonical complex number systems corresponding to $i^2 = -1$, 0 and $+1$. Following Hucks [13], these will be referred to here as standard (Euclidean), Galilean, and hyperbolic (complex) numbers respectively. In the case of Galilean complex numbers [14] the law of multiplication of two numbers is:

$$(a + bi)(c + di) = ac + (ad + bc)i, \tag{1}$$

and in the case of hyperbolic complex numbers the law takes the form:

$$(a + bi)(c + di) = (ac + bd) + (ad + bc)i. \tag{2}$$

An important point here is that the concept of analytic (or holomorphic) function as a function which depends only on $z = a + bi$ but not on $z^* = a - bi$ can be naturally extended to the case of Galilean and hyperbolic numbers. The Cauchy–Riemann equations as necessary conditions of analyticity can also be easily formulated.

It can be noted that hyperbolic complex numbers seem to be particularly interesting from the point of view of theoretical physics. Indeed, a reasonably far-reaching theory of analytic functions can be built based on their set. The corresponding Cauchy–Riemann equations are no more than the free Maxwell equations in $(1 + 1)$ dimensions. Taking into account that every analytic function provides solutions to the Cauchy–Riemann equations, solutions to $(1 + 1)$-dimensional Maxwell equations can be obtained, which might satisfy interesting boundary conditions [13]. This could be of interest, for example, from the point of view of cavity quantum electrodynamics with moving boundaries where exact oscillating solutions to the Maxwell equations are important (see, e.g. [15–17]).

3. Results

In the first part of this section, several pictures of biomorphs obtained from the iterations of almost the simplest possible analytic functions, namely quadratic and cubic ones, are presented. The second part is devoted to their analysis using well-defined statistical and complexity measures. In the latter part, the study has been deliberately restricted to such simple functions in order to avoid dealing with too much scattered data with the loss of necessary focus.

3.1. Standard Pickover biomorphs from iteration of a cubic function

Let us consider the following iteration:

$$z \longleftarrow z^n + C, \tag{3}$$

where $\longleftarrow$ is the assignment operator (as in [18]) and we then introduce the following criterion to stop the iteration procedure:

$$Re(z)^2 > r_0 \quad \text{or} \quad Im(z)^2 > r_0. \tag{4}$$

It is precisely because of the seemingly innocent "or" statement in the above criterion that the images produced by the appropriate colouring of the structure obtained differ radically from otherwise closely related Mandelbrot or Julia sets. Examples of standard Pickover biomorphs obtained from the above procedure with $n = 2$ and $n = 3$ are shown in Fig. 1. The "simulation cell" has a dimension of 1024×1024 pixels and the starting points have been located in the square $-2.5 \leq Re(z) \leq 2.5$, $-2.5 \leq Im(z) \leq 2.5$. In all examples shown below the maximal number of iterations was equal to 100.

The value of r_0 has been set equal to 10.0 and C has been set equal to 0.5 (i.e. with a zero imaginary part).

3.2. Pickover biomorphs in the realm of Galilean numbers

The same quadratic and cubic iteration schemes and parameters are applied to Galilean complex numbers and result in the image shown in Fig. 2:

3.3. Pickover biomorphs and hyperbolic complex numbers

If it is now assumed that $i^2 = +1$, the Pickover biomorphs take very different forms as illustrated in Fig. 3.

The self-similarity though evident is clearly restricted and similar structures seem to be shifted and spatially separated from each other.

3.4. Pickover biomorphs for the squares of imaginary units dependent on points in the complex plane

The study has also experimented with a type of complex numbers with the following multiplication law:

$$(a + bi)(c + di) = (ac + bdi^2(a, b, c, d)) + (ad + bc)i, \tag{5}$$

with the obvious requirement of symmetry $i^2(a, b, c, d) = i^2(c, d, a, b)$. The following images have been obtained from the iterations of a cubic function and three different Gaussian forms of i^2 depending only on a and c. On the left-hand side of this figure $i^2 = -1 + \exp(-2(a^2 + c^2))$, at the centre $i^2 = \exp(-2(a^2 + c^2))$, and on the right-hand side $i^2 = -1 + 2\exp(-2(a^2 + c^2))$ (Fig. 4).

3.4.1. Correlations in images of Pickover biomorphs

As a measure of correlations in the images of Pickover biomorphs Moran's autocorrelation function I [19] has been chosen and is defined as:

$$I = \frac{N}{W} \frac{\sum_{i,j} w_{ij}(x_i - \bar{x})(x_j - \bar{x})}{\sum_i (x_i - \bar{x})^2}, \tag{6}$$

where N is the number of spatial units indexed by i and j, $\bar{x}$ is the mean of all $\{x_i\}$, w_{ij} is a matrix of weights such that $w_{ii} = 0$, and $W = \sum_{i,j} w_{ij}$. It is well known that the value of I may strongly depend on the choice of matrix of weights. An attempt has not been made to obtain any comprehensive picture of such dependence in our case, but instead a simple binary matrix has been applied. Firstly, w_{ij} was chosen to be 1 for the four nearest neighbours and 0 otherwise. Secondly, the computations were then repeated for $w_{ij} = 1$ for the six nearest neighbours. These choices of neighbours correspond to the von Neumann and Moore neighbourhoods in cellular automata theory [20]. The biomorph for analysis was obtained using a grey-scale picture of the size of 1024×1024 pixels.

It appears that the correlations in biomorphs do not depend much on the system of weights in the Moran autocorrelation function. Furthermore, the qualitative outcome of the dependence on i^2 is the same for quadratic and cubic functions. We have observed in addition that Moran's function grows when the size of the pictures is increased from 128×128 to 1024×1024 (Figs. 5 and 6).