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Fractal interpolation functions for random data sets

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ABSTRACT

Let $x_0 < x_1 < x_2 < \ldots < x_N$ and $I = [x_0, x_N]$. Let u be a continuous function defined on I and let $\Delta_\mu = \{(x_k, \mu_k) : k = 0, 1, \ldots, N\}$, where $\mu_k = u(x_k)$. We establish a fractal interpolation function $f_{\langle T_\mu \rangle}$ on I corresponding to the set of points Δ_μ. Let Y_k be a random perturbation of μ_k and set $\Delta_Y = \{(x_k, Y_k) : k = 0, 1, \ldots, N\}$. By a similar way, we construct a fractal interpolation function $f_{\langle T_Y \rangle}$ on I corresponding to the set Δ_Y. $f_{\langle T_Y \rangle}(x)$ is a random variable for any $x \in I$, and the function $f_{\langle T_Y \rangle}$ can be treated as a fractal perturbation of u under some random noise in the set of interpolation points Δ_μ. In this article we investigate some statistical properties of $f_{\langle T_Y \rangle}$ and give estimations of the difference between $f_{\langle T_Y \rangle}$ and u.

1. Introduction

A fractal function is a function whose graph is the attractor of an iterated function system and a fractal interpolation function (FIF) is a continuous fractal function interpolating a given set of points. FIFs are the basis of a constructive approximation theory for non differentiable functions. The concept of FIFs was introduced in [1] and various types of FIFs have been constructed in different ways, see [3,5,7,9–11,15,24,25,28] and the references therein. Many approximation properties of FIFs are discussed in [16,17,21–23,25–28].

Here we give a brief introduction to the construction of a FIF. The readers are referred to [1,2,25] for more details. Consider a given set of points $\Delta_\mu = \{(x_k, \mu_k) \in \mathbb{R} \times \mathbb{R} : k = 0, 1, \ldots, N\}$, where $x_0 < x_1 < x_2 < \ldots < x_N$. Let $I = [x_0, x_N]$ and $I_k = [x_{k-1}, x_k]$ for each $k = 1, \ldots, N$. Let L_k: $I \to I_k$ be a contractive homeomorphism such that $L_k(x_0) = x_{k-1}$ and $L_k(x_N) = x_k$. Choose a large enough compact interval D in $\mathbb{R}$ and let $K = I \times D$. For $k = 1, \ldots, N$, define M_k: $K \to D$ by

$$M_k(x, y) = \alpha_k y + q_k(x), \qquad \text{where} \quad -1 < \alpha_k < 1. \tag{1.1}$$

Here $q_k : I \to \mathbb{R}$ is a function such that $M_k(x_0, \mu_0) = \mu_{k-1}$ and $M_k(x_N, \mu_N) = \mu_k$. Let $W_k(x, y) = (L_k(x), M_k(x, y))$. Then there is exactly one attractor $G \subseteq K$ of the iterated function system $\mathcal{I} = \{K; W_1, \ldots, W_N\}$, and moreover, G is the graph of a continuous function $g : I \to \mathbb{R}$ which satisfies $g(x_k) = \mu_k$ for $k = 0, \ldots, N$. The function g is called a fractal interpolation function (FIF) corresponding to the system $\mathcal{I}$ and the set of points Δ_μ, and g also satisfies the functional equation

$$g(x) = \alpha_k g(L_k^{-1}(x)) + q_k(L_k^{-1}(x)), \quad x \in I_k. \tag{1.2}$$

If each q_k is a polynomial, then g is termed a polynomial FIF. If each q_k is a rational function, then g is called a rational FIF [25]. Consider the form

$$q_k(x) = u(L_k(x)) - \alpha_k r(x), \tag{1.3}$$

where u and r are continuous functions on I such that $u(x_k) = \mu_k$ for all $k = 0, \ldots, N$ and $r(x_0) = \mu_0$, $r(x_N) = \mu_N$. In [1] the functions u and r are called the height function and the base function, respectively. In the case $r = L(u)$, where L is a bounded linear operator on the space of all continuous functions defined on I, the corresponding FIF g is usually called α-fractal function associated with u [21] and is denoted by u^α, where $\alpha = (\alpha_1, \ldots, \alpha_N)$ is a vector. u^α can be treated as a fractal perturbation of u. The operator $\mathcal{F}^\alpha : u \mapsto u^\alpha$ is called an α-fractal operator. The developments of properties of $\mathcal{F}^\alpha$ delineated a theory which is referred to as fractal approximation theory, see for instance [22,23,25].

A class of FIFs which are established by iterated function systems was considered in [28], where each M_k is given by (1.1) and (1.3) with all the constants α_k being replaced by variable parameters $\alpha_k(x)$, where $\alpha_k(x)$ is a Lipschitz function defined on I such that $\sup_{x \in I} |\alpha_k(x)| < 1$. The authors studied the stability and sensitivity of FIFs with variable parameters. In particular, an estimation of $\|f_{\langle T_Y \rangle} - f_{\langle T_\mu \rangle}\|_\infty$ is given in [28, Theorem 4.1], where $f_{\langle T_\mu \rangle}$ is the FIF determined by such an iterated function system,

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u is the piecewise linear interpolation function through the set of points Δ_μ, and r is the linear function through the points (x_0, μ_0) and (x_N, μ_N). On the other hand, $f_{\langle T_\mathcal{Y}\rangle}$ is the FIF determined by the iterated function system with functions $\mathcal{M}_k(x,y) = \alpha_k(x)y + v(L_k(x)) - \alpha_k(x)\tau(x),\ k = 1, \ldots, N,$ where v is the piecewise linear interpolation function through the set of points $\Delta_\mathcal{Y} = \{(x_k, y_k) \in \mathbb{R}\times\mathbb{R} : k = 0, 1, \ldots, N\}$, and τ is the linear function through the points (x_0, y_0) and (x_N, y_N).

In the construction of α-fractal functions mentioned above, the domains of functions L_k and M_k are I and $I \times D$, respectively, where $I = [x_0, x_N]$. In the articles [4,8], a recurrent structure for iterated function systems was introduced and recurrent FIFs were obtained by such systems. In the construction of recurrent FIFs, the domains of L_k and M_k are constrained to $J_{\mathbb{J}(k)}$ and $J_{\mathbb{J}(k)} \times D$, respectively, where $J_{\mathbb{J}(k)} = [\hat{x}_{\mathbb{J}(k)-1}, \hat{x}_{\mathbb{J}(k)}]$, $\hat{x}_{\mathbb{J}(k)-1}, \hat{x}_{\mathbb{J}(k)} \in \{x_0, \ldots, x_N\}$ and $\hat{x}_{\mathbb{J}(k)-1} < \hat{x}_{\mathbb{J}(k)}$. In this paper, we establish FIFs based on the ideas of the construction of α-fractal functions and recurrent FIFs. The domains of L_k and M_k are J_k and $J_k \times \mathbb{R}$, respectively, where $J_k = [x_{j(k)}, x_{l(k)}]$ is a subinterval of I which depends on k.

In regression analysis and signal analysis, statistical models are established by observed data, and these data are supposed to be random. It is also known that interpolation techniques have been widely used in the reconstruction of random signals from samples [6,12,14,20]. The theory of iterated function systems and fractal interpolation functions have been applied to model discrete sequences [13,18,19]. The problem we considered here is motivated by these literatures. In this paper, we investigate some statistical properties of fractal interpolation functions corresponding to a set of data, where these data are supposed to be random.

Throughout this paper, let N be an integer greater or equal to 2 and let $x_0 < x_1 < x_2 < \ldots < x_N$. Denote that I is the interval $[x_0, x_N]$ and $C[I]$ is the set of all real-valued continuous functions defined on I. Let $u \in C[I]$ and $u(x_k) = \mu_k$ for all $k = 0, \ldots, N$. We first construct a FIF $f_{\langle T_\mu\rangle}$ corresponding to the set of points $\Delta_\mu = \{(x_k, \mu_k) : k = 0, 1, \ldots, N\}$. Then we consider each y_k as an observed value of a random perturbation of μ_k. If each y_k is treated as a random variable Y_k, then by a similar process, we have an analog FIF $f_{\langle T_Y\rangle}$ corresponding to the set $\Delta_Y = \{(x_k, Y_k) : k = 0, 1, \ldots, N\}$. All values $f_{\langle T_Y\rangle}(x),\ x \in I$, are random. $f_{\langle T_Y\rangle}$ can be treated as a fractal perturbation of u under some random noise in the set of interpolation points Δ_μ. The purpose of this paper is to investigate some statistical properties of $f_{\langle T_Y\rangle}(x)$ and establish estimations of $f_{\langle T_Y\rangle} - f_{\langle T_\mu\rangle}$ and $f_{\langle T_Y\rangle} - u$.

Define $\|f\|_\infty = \max_{x\in I} |f(x)|$ and $\|f\|_p = (\int_I |f(x)|^p dx)^{1/p}$ for $f \in C[I]$, where $1 \le p < \infty$. Let $\mathbf{E}(\cdot)$ and $\mathbf{Var}(\cdot)$ be the expectation and the variance of a random variable, respectively. If $g(x)$ is a random variable for all $x \in I$, we define $\|\mathbf{E}(g)\|_\infty = \sup_{x\in I} |\mathbf{E}(g(x))|$. For a given set of points $\Delta = \{(x_k, z_k) : k = 0, 1, \ldots, N\}$, let $C_\Delta[I]$ be the set of functions in $C[I]$ that interpolate all points in Δ. It is known that $(C[I], \|\cdot\|_\infty)$ is a Banach space and $C_\Delta[I]$ is a complete metric space, where the metric is induced by $\|\cdot\|_\infty$.

2. Construction of fractal interpolation functions

For $k = 1, \ldots, N$, let $I_k = [x_{k-1}, x_k]$ and $J_k = [x_{j(k)}, x_{l(k)}]$, where $j(k), l(k) \in \{0, 1, \ldots, N\}$ and $j(k) < l(k)$. For each k, $x_{j(k)}, x_{l(k)} \in \{x_0, x_1, \ldots, x_N\}$ and J_k is a subinterval of I with endpoints x_i, x_m for some $0 \le i < m \le N$. To avoid trivial cases, we assume that $J_k \ne I_k$. We introduce functions L_k and α_k for $k = 1, \ldots, N$ as follows.

(I) L_k: $J_k \to I_k$ is a homeomorphism such that $L_k(x_{j(k)}) = x_{k-1}$, $L_k(x_{l(k)}) = x_k$.
(II) $\alpha_k : J_k \to \mathbb{R}$ is a continuous function such that $|\alpha_k(x)| \le s_k$ for all $x \in J_k$ and for some $s_k > 0$.

Let $u \in C[I]$ and $\mu_k = u(x_k)$ for $k = 0, 1, \ldots, N$. Let $\Delta_\mu = \{(x_k, \mu_k) : k = 0, 1, \ldots, N\}$ and we suppose that all the data points in Δ_μ are non-collinear. Let $\mathcal{Q}_\mu = \{(x_{j(k)}, \mu_{j(k)}), (x_{l(k)}, \mu_{l(k)}) : k = 1, \ldots, N\}$ be a subset of Δ_μ. Define $M_k : J_k \times \mathbb{R} \to \mathbb{R}$ by

$$M_k(x, y) = \alpha_k(x)y + u(L_k(x)) - \alpha_k(x)\gamma(x), \tag{2.1}$$

where $\gamma \in C_{\mathcal{Q}_\mu}[I]$. Then $M_k(x_{j(k)}, \mu_{j(k)}) = \mu_{k-1},\ M_k(x_{l(k)}, \mu_{l(k)}) = \mu_k$, and

$$|M_k(x, y) - M_k(x, y^*)| \le s_k|y - y^*| \text{ for all } x \in J_k \text{ and } y, y^* \in \mathbb{R}. \tag{2.2}$$

Define $W_k : J_k \times \mathbb{R} \to I_k \times \mathbb{R}$ by

$$W_k(x, y) = (L_k(x), M_k(x, y)),\ x \in J_k,\ y \in \mathbb{R}.$$

For $h \in C_{\Delta_\mu}[I]$ and for each $k = 1, \ldots, N$, define $A_k = \{(x, h(x)) : x \in J_k\}$. Then

$$W_k(A_k) = \{(L_k(x), M_k(x, h(x))) : x \in J_k\}. \tag{2.3}$$

Since L_k: $J_k \to I_k$ is a homeomorphism, $W_k(A_k)$ in (2.3) can be written as

$$W_k(A_k) = \{(x, M_k(L_k^{-1}(x), h(L_k^{-1}(x)))) : x \in I_k\}. \tag{2.4}$$

Note that A_k is the subgraph of h on J_k and W_k maps this graph to the region $I_k \times \mathbb{R}$. Moreover, $W_k(A_k)$ is the graph of the continuous function $h_k : I_k \to \mathbb{R}$ defined by

$$h_k(x) = M_k(L_k^{-1}(x), h(L_k^{-1}(x))), \quad x \in I_k. \tag{2.5}$$

It is easy to see that $h_k(x_{k-1}) = \mu_{k-1}$ and $h_k(x_k) = \mu_k$. Define a mapping $T_\mu : C_{\Delta_\mu}[I] \to C_{\Delta_\mu}[I]$ by $T_\mu(h)(x) = h_k(x)$ for $x \in I_k$, that is, for $h \in C_{\Delta_\mu}[I]$ and $x \in I_k$,

$$T_\mu(h)(x) = \alpha_k(L_k^{-1}(x))h(L_k^{-1}(x)) + u(x) - \alpha_k(L_k^{-1}(x))\gamma(L_k^{-1}(x)). \tag{2.6}$$

Define

$$s = \max\{s_1, \ldots, s_N\}.$$

We have the following theorem.

Theorem 2.1. *If $0 < s < 1$, then the operator T_μ defined by (2.6) is a contraction mapping on $C_{\Delta_\mu}[I]$.*

Proof. Let $h_1, h_2 \in C_{\Delta_\mu}[I]$. By (2.6) and (II) we see that, for $x \in I_k$,

$$|T_\mu(h_1)(x) - T_\mu(h_2)(x)| \le s_k|h_1(L_k^{-1}(x)) - h_2(L_k^{-1}(x))|.$$

Then by (I),

$$\begin{aligned}\|T_\mu(h_1) - T_\mu(h_2)\|_\infty &\le \max_{k=1,\ldots,N} s_k\left\{\max_{z\in J_k} |h_1(z) - h_2(z)|\right\} \\ &\le s\|h_1 - h_2\|_\infty.\end{aligned} \tag{2.7}$$

The condition $0 < s < 1$ implies that T_μ is a contraction mapping on $C_{\Delta_\mu}[I]$. □

Definition 2.2. The fixed point $f_{\langle T_\mu\rangle}$ of T_μ in $C_{\Delta_\mu}[I]$ is called a fractal interpolation function (FIF) on I corresponding to the set of points Δ_μ.

The FIF $f_{\langle T_\mu\rangle}$ given in Definition 2.2 satisfies the equation for $k = 1, \ldots, N$:

$$f_{\langle T_\mu\rangle}(x) = \alpha_k(L_k^{-1}(x))\left\{f_{\langle T_\mu\rangle}(L_k^{-1}(x)) - \gamma(L_k^{-1}(x))\right\} + u(x),\ x \in I_k. \tag{2.8}$$

Remark 2.3. In general, to construct a fractal interpolation function $f_{\langle T_\mu\rangle}$, we can replace the base function $\gamma(x)$ in (2.1) by $\gamma_k(x)$, where γ_k is a continuous function defined on J_k such that $\gamma_k(x_{j(k)}) = \mu_{j(k)}$ and $\gamma_k(x_{l(k)}) = \mu_{l(k)}$. Then we define T_μ by (2.6) with γ being replaced by γ_k. Theorem 2.1 still holds in this