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Electrostatic interactions of arbitrarily dispersed multicoated elliptic cylinders

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ABSTRACT

We propose a theoretical framework for evaluation of electrostatic potentials in an unbounded isotropic matrix containing a number of arbitrarily dispersed elliptic cylinders subjected to a remotely prescribed potential field. The inclusions could be homogeneous or confocally multicoated, and may have different sizes, aspect ratios and different conductivities. The approach is based on a multipole expansion formalism, together with a construction of consistency conditions and translation operators. This procedure generalizes the approach of the classic work of Rayleigh [1] for a periodic array of circular disks or spheres to an arbitrarily dispersion of elliptic cylinders. We combine the methods of complex potentials with a re-expansion formulae and the generalized Rayleigh's formualtion to obtain a complete solution of the many-inclusion problem. We show that the coefficients of field expansions can be written in the form of an infinite set of linear algebraic equations. Numerical results are presented for several configuarions. We further apply the obtained field solutions to determine the effective conductivity of the composite.

1. Introduction

The transport phenomena of a heterogeneous medium consisting of many inclusions embedded in a host matrix are of fundamental theoretical interest, and also play an important role in optimal designs of industrial products. Among numerous aspects on micromechanics, most of the studies focused on the overall properties of composite materials. In contrast, the study on the field solutions appears to be a relatively rarely explored area due to its complexity. However, local fields of a composite medium are also of theoretical and technological interests in interpreting various physical phenomena such as the breakdown phenomenon [2] and in higher-order estimates of the overall properties [3]. Further, much work has assumed that the inclusions are circular or spherical, and that each phase possesses a constant property. In real systems, however, inclusions are not perfect circles or spheres, and there are situations in which the inclusions are inhomogeneous. For example, in some applications, to reduce heat or stress concentration along the interface, interphase layers between the inclusions and the matrix are often introduced to act as thermal barriers. Such interface layer may be continuous graded transition in composition or in fine, discrete step, across the interface between the two dissimilar materials. Materials with spatially varying properties, referred to as functionally graded materials, have found merits in quite a few engineering applications [4]. The main characteristic of spatially varying materials is the tailoring of graded composition to satisfy particular engineering applications. For a good review of research on functionally graded materials, the reader is referred to Suresh [4] and Hirai [5].

The present paper is concerned with the electrostatic potential field of an unbounded isotropic medium containing arbitrarily dispersed graded elliptic cylinders. Mathematically, the frameworks of electrostatics, dielectrics, magnetism, thermal

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conduction, diffusion or fluid transport, and anti-plane elasticity are all equivalent ([6, p. 19]). The solutions of any one physical problem can be readily applicable to others. Here we use the terminology of electrostatics throughout the paper simply for convenience. We consider the case that the elliptic fibers have different sizes, aspect ratios with different conductivities, and are arbitrarily positioned in the unbounded space or in a periodic array. Throughout the formulation we allow that the cylinders are homogeneous elliptic fibers or confocally multicoated ellipses. The framework of this study is based on the concept of multipole expansion formalism, together with a construction of consistency conditions and translation operators. The procedure generalizes the approach of the classic work of Rayleigh [1] for periodic arrays of circular cylinders and spheres to a medium containing many multicoated elliptic cylinders that can be arbitrarily positioned. We have employed the methods of complex potentials, a newly derived re-expansion formulae by Kushch et al. [7], and the generalized Rayleigh's formulation [8,9] to calculate the field potential and the effective conductivity of the considered composite.

The Rayleigh's formulation has been used in previous work of Nicrovici and McPhedran [10] and Yardley et al. [11] to determine the effective transport property of a rectangular array of elliptic cylinders. Yardley et al. [12] employed integral transforms to obtain rapidly convergent series for one-dimensional elliptical lattice sums, and used these lattice sums to determine the transport property of composites constructed from multiple layers of elliptical cylinders. Parallel studies have been carried out for elasticity by other methods. Meisner and Kouris [13] used the Papkovich–Neuber displacement approach to study the interaction of two elliptic inclusions under a far-field biaxial tension. Kushch et al. [7] combined standard Muskhelishvili's representation of general solution in terms of complex potentials, the superposition principle and re-expansion formulae to obtain a complete solution of the many-inclusion problem. This analytical approach has been further developed and applied for studying the elastic behavior of a half plane containing a finite array of elliptic inclusions [14], and for the local stress and the effective elastic properties of unidirectional fiber reinforced composite with anisotropic phases [15]. For composites reinforced by discrete graded elliptic fibers, Ru et al. [16] showed that a three-phase elliptic inclusion under uniform remote stress and eigenstrain in anti-plane shear admits internal uniform stress field provided that the interfaces are two confocal ellipses. Ting et al. [17] provided a theoretical framework showing how to design a neutral cylinder with any number of coatings or with graded shear moduli in a cross-section under torsion. Other works included that of Chen [18] who considered the anti-plane shear of composites with a confocally multicoated elliptical inclusion. To our knowledge, the subject of composites with arbitrarily dispersed multicoated elliptic cylinders has not been examined in the literature before.

The plan of the paper is as follows. In Section 2, we outline the framework of the electrostatic problem of an unbounded medium containing a number of homogeneous isotropic elliptic cylinders. The potential field is expanded versus various local elliptic coordinates with origins positioned at each inclusion's centroid. The key step is to link the potentials with the remote applied field. This is accomplished by making use of Green's second identity in the matrix domain. In Section 3 we consider the case of multicoated elliptic cylinders. We propose a recurrence procedure analogous to a method originally devised for a multicoated circular cylinder [17]. We show that a (2×2) array alone can mathematically simulate the effects of multiple coatings. In Section 4, we determine the effective conductivity from the obtained field solutions, and show that the overall property solely depends on one coefficient B_1 among the infinite number of expansion coefficients. We show in Section 5 that coefficients of the field expansions is governed by an infinite set of linear algebraic equation. Numerical results are presented for several configurations, and the effects of multiple coatings is exemplified numerically. Lastly, some concluding remarks are made in Section 6.

2. Multiple elliptic cylinders

2.1. Problem statement

We consider an infinite medium Ω containing N arbitrarily distributed elliptic cylinders. The domain of the pth elliptic cylinder is represented by V_p, $p = 1, 2, \ldots, N$, each of which has the conductivity tensor $\mathbf{k}_p$. The remaining part, the matrix Ω_m, is isotropic with conductivity $k_m\mathbf{I}$. Let us introduce a Cartesian coordinate system (x, y), or symbolically as $\mathbf{x}$, positioned at a selected point O of the plane (Fig. 1). The centroids of the pth elliptic cylinders are designated as O_p, with $O_p x_p$ and $O_p y_p$ axes directed along the major and minor axes of the ellipse. Each of ellipse has the major and minor semi-axis, $l_x^{(p)}$ and $l_y^{(p)}$, and the inter-foci distance is $2d_p$, where $d_p^2 = l_x^{(p)2} - l_y^{(p)2}$. The ellipses are all well separated so that any two inclusions will not get in touch with each other. Along the remote boundary of the matrix, an external potential field is prescribed. Under steady-state condition, the potential field φ in the medium is governed by

$$\nabla \cdot [\mathbf{k}(\mathbf{x}) \nabla \varphi] = 0 \quad \text{in } \Omega \tag{2.1}$$

with

$$\mathbf{k}(\mathbf{x}) = \begin{cases} k_m \mathbf{I} & \text{if } \mathbf{x} \in \Omega_m, \\ \mathbf{k}_p & \text{if } \mathbf{x} \in V_p. \end{cases} \tag{2.2}$$

Here $\mathbf{k}_p$ may vary with $\mathbf{x}$, and $\mathbf{I}$ is the identity matrix. For an isotropic phase, the governing field (2.1) for the potential is simply the Laplace equation. In the analysis we assume that the inclusion and the matrix are perfectly bonded at their interfaces ∂V_p. This means that the electrostatic potential and the normal component of electric current are continuous across the interfaces