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A stochastic viral infection model driven by Lévy noise

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ABSTRACT

In this paper, we are interested in the study of a stochastic viral infection model with immune impairment driven by Lévy noise. First we prove the existence of a unique global solution to the model. By means of the Lyapunov method we study the stability of the equilibria. We present sufficient conditions for the extinction and persistence in mean. Furthermore, we present some numerical results to support the theoretical work.

1. Introduction and formulation of the model

Infectious diseases are still presented as one of the serious threats to human lives. Understanding the way of the spread of those diseases is very important in the fight against the disease itself [2,3,7,14]. Mathematical modelling plays a significant role as an important tool used in describing and understanding the spread of infections, and it is one major area of biology. Viral infections represent a dangerous infectious diseases. When it occurs, a viral infection can be initially rapid and nonspecific (natural killer cells, macrophage cells, etc.) and then delayed and specific (cytotoxic T lymphocyte cells, antibody cell). Likely, in most virus infections, cytotoxic T lymphocyte cells (CTL) which attack infected cells, and antibody cells which attack viruses, play a critical role in antiviral defense. In order to investigate the role of the population dynamics of viral infection with CTL response, Regoes et al. [16] and Wang et al. [17] established a mathematical model describing the basic dynamics of the interaction between activated CD4+ T cells $x(t)$, infected CD4+ T cells $y(t)$, and immune cells $z(t)$. That model is presented by

$$\begin{aligned} dx &= (s - \mu x - \beta xy)dt \\ dy &= (\beta xy - ay - pyz)dt \\ dz &= (cy - bz - myz)dt, \end{aligned} \tag{1.1}$$

where activated CD4+ T cells are produced at a rate of s cells day^{-1}, decay at a rate μ day^{-1}, and can become infected at a rate that is proportional to the number of infected CD4+ T cells $y(t)$ with a transmission rate constant β day^{-1} cell^{-1}. The infected CD4+ T cells are assumed to decay at the rate of a day^{-1}. The CTL responses eliminate at a rate that is proportional to the number of CTLs with a killing rate constant p day^{-1} cell^{-1}, proliferate at the rate of c day^{-1} and decay at a rate of b day^{-1} and the immune impairment rate m day^{-1} cell^{-1}.

The model (1.1) has a reproduction number which is given by $\mathcal{R}_0 = \frac{\beta s}{a\mu}$, an uninfected equilibrium $E^0 = (\frac{s}{\mu}, 0, 0)$ and an infected equilibrium $E^* = (x^*, y^*, z^*)$ such that

$$x^* = \frac{1}{\beta}\left(a + \frac{cpy^*}{b + my^*}\right),\ y^* = \frac{1}{2A}\left(-B + \sqrt{B^2 + 4Ab(s\beta - a\mu)}\right),$$
$$z^* = \frac{cy^*}{b + my^*},$$

where $A = \beta(am + cp)$ and $B = ab\beta + c\mu p - m(s\beta - a\mu)$. In the last years, authors have been paying a great interest to models with stochastic perturbation [9,10,12,13,23], due to the fact that stochastic models, take into account not only the mean trend but also the variance structure around it. For fixed starting values, a deterministic model will always produce the same result whereas a stochastic model may produce many different outputs, depending on the actual values the random variables take. Recently, Rajaji and Pitchaimani [15], studied the following stochastic viral infection model

$$\begin{aligned} dx &= (s - \mu x - \beta xy)dt - \sigma xydW \\ dy &= (\beta xy - ay - pyz)dt + \sigma xydW \\ dz &= (cy - bz - myz)dt, \end{aligned} \tag{1.2}$$

where σ is the intensity of the Brownian motion W defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the filtration $(\mathcal{F}_t)_{t\geq 0}$,

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satisfying the usual conditions. The authors showed that this model has a unique global solution, using the Lyapunov method. They investigated the stochastic stability of equilibria points and they presented a sufficient condition for persistence of the disease. The system (1.2) is a stochastic model driven only by white noise, therefore its solution is continuous. However, when encountered with massive diseases like avian influenza, such a disturbance may break the continuity of the solution. Thus the importance of Lévy noise in the study of the dynamical behavior of the model, is of great significance to the prevention and the control of the disease. For recent works on dynamics of solution to a stochastic model driven by a Lévy noise, see [4–6,18–22]. In this work we are interested in investigating stochastic viral infection model with immune impairment driven by Lévy noise. Such model can be presented by

$$\begin{aligned} dx &= (s-\mu x-\beta xy)dt-\sigma xydW-\int_A Q(\alpha)x(t-)y(t-)\tilde{N}(dt,d\alpha)\\ dy &= (\beta xy-ay-pyz)dt+\sigma xydW+\int_A Q(\alpha)x(t-)y(t-)\tilde{N}(dt,d\alpha)\\ dz &= (cy-bz-myz)dt. \end{aligned} \tag{1.3}$$

On the foundation of the models (1.1) and (1.2), the system (1.3) has been linked to a stochastic perturbation, in which $X(t-)$ is the left limit of $X(t)$, $N(dt,\, d\alpha)$ is a Poisson counting measure with the stationary compensator $\nu(d\alpha)dt$, $\tilde{N}(dt,d\alpha)=N(dt,d\alpha)-\nu(d\alpha)dt$ and ν is defined on a measurable subset A of $[0,\infty)$ with $\nu(A)<\infty$ and $Q(\alpha)>-1$. Our study will be as follow: in the second section we establish the existence of a unique global solution to the system (1.3), in the third section we study the extinction of the disease, the forth part is devoted to the study of the behavior of the solution to (1.3) around the infected equilibrium E^*, in the last part we investigate the persistence of the disease.

2. Global solution of (1.3)

In this section, we will establish the existence of a unique global positive solution for our stochastic epidemic model with relapse and jumps. In what follows, we shall impose two standard assumptions, **(H1)** and **(H2)**, which are essential to prove the existence and uniqueness of a global positive solution of (1.3).

(H1) For each $M>0$ there exists $L_M>0$ such that

$\int_A |Z(x_1,y)-Z(x_2,y)|^2\nu(d\alpha)\le L_M|x_1-x_2|^2 \quad i=1,2,$ with $|x_1|\vee|x_2|\le M$, where

$$Z(x,y)=Q(\alpha)xy.$$

(H2) $|\log(1+Q(\alpha))|\le C$, where C is a positive constant.

Define

$$\mathbb{D}=\left\{(x,y,z)\in\mathbb{R}^3;\ x>0,\ y>0,\ z>0,\ \ x+y+\frac{a}{2c}z\le\frac{s}{q}\right\},$$

where $q=\min\{\mu,\frac{a}{2},b\}$.

Theorem 2.1. *For any initial condition $(x(0),y(0),z(0))\in\mathbb{D}$, there exists a unique positive solution $(x(t),y(t),z(t))\in\mathbb{D}\ \forall t\ge 0$ a.s.*

Proof. Since the drift and the diffusion are locally Lipschitz, by **(H1)** then for any $(x(0),y(0),z(0))\in\mathbb{D}$ there exists a unique local solution for $t\in[0,\ \tau_e)$, where τ_e is the explosion time. First we show that $x(t)+y(t)+\frac{a}{2c}z(t)\le\frac{s}{q}\ \forall t\in[0,\tau_e)$. Let $L=x+y+\frac{a}{2c}z$ and $q=\min\{\mu,\frac{a}{2},b\}$. We have $L'=s-\mu x-\frac{a}{2}y-\frac{ab}{2c}z-(p+\frac{am}{2c})yz<s-qL$. Then $L<\frac{s}{q},\ \forall t\in[0,\tau_e)$ a.s. Next we show that $\tau_e=\infty$ a.s. Define the stopping time

$$\tau=\inf\{t\in[0,\tau_e),\ x(t)\le 0\ \ or\ \ y(t)\le 0\ \ or\ \ z(t)\le 0\},$$

we have $\tau\le\tau_e$. We assume that $\tau<\infty$. Then $\exists T>0$ such that $\mathbb{P}\{\tau\le T\}>0$. We consider the following function

$$V(x,y,z)=\log(xyz). \tag{2.1}$$

Applying Its formula yields

$$\begin{aligned} dV &= \mathcal{L}V\,dt+\sigma(x-y)dW\\ &\quad+\int_A\log((1-Q(\alpha)y)(1+Q(\alpha)x))\tilde{N}(dt,d\alpha) \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} \mathcal{L}V &= \frac{1}{x}(s-\mu x-\beta xy)+\frac{1}{y}(\beta xy-ay-pyz)\\ &\quad+\frac{1}{z}(cy-bz-myz)-\frac{\sigma^2y^2}{2}-\frac{\sigma^2x^2}{2}\\ &\quad+\int_A Q(\alpha)y+\log(1-Q(\alpha)y)\nu(d\alpha)\\ &\quad-\int_A Q(\alpha)x-\log(1+Q(\alpha)x)\nu(d\alpha). \end{aligned} \tag{2.3}$$

Then

$$\begin{aligned} \mathcal{L}V &= \frac{s}{x}-\mu-\beta y+\beta x-a-pz+c\frac{y}{z}-b-my-\frac{\sigma^2y^2}{2}-\frac{\sigma^2x^2}{2}\\ &\quad+\int_A Q(\alpha)y+\log(1-Q(\alpha)y)\nu(d\alpha)\\ &\quad-\int_A Q(\alpha)x-\log(1+Q(\alpha)x)\nu(d\alpha). \end{aligned}$$

Since $x+y+\frac{2}{2c}z\le\frac{s}{q}$, we have

$$\begin{aligned} \mathcal{L}V &\ge -\mu-a-b-(\beta+\frac{2pc}{a}+m)\frac{s}{q}-\sigma^2\frac{s^2}{q^2}\\ &\quad-2\int_A\log(1+Q(\alpha)\frac{s}{q})\nu(d\alpha):=K. \end{aligned}$$

Therefore

$$\begin{aligned} dV &\ge Kdt+\sigma(x-y)dW\\ &\quad+\int_A\log((1-Q(\alpha)y)(1+Q(\alpha)x))\tilde{N}(dt,d\alpha). \end{aligned} \tag{2.4}$$

Integrating both sides of (2.4) between 0 and t we get

$$\begin{aligned} &V(x(t),y(t),z(t))\\ &\ge V(x(0),y(0),z(0))+Kt+\sigma\int_0^t(x-y)dW\\ &\quad+\int_0^t\int_A\log((1-Q(\alpha)y)(1+Q(\alpha)x))\tilde{N}(dt,d\alpha). \end{aligned} \tag{2.5}$$

Some components of $(x(\tau),y(\tau),z(\tau))$ being equal to 0, we obtain

$$\lim_{t\to\tau}V(x,y,z)=-\infty.$$

Then letting $t\to\tau$ in (2.5) we have

$$\begin{aligned} -\infty &\ge V(x(0),y(0),z(0))+K\tau+\sigma\int_0^\tau(x-y)dW\\ &\quad+\int_0^\tau\int_A\log((1-Q(\alpha)y)(1+Q(\alpha)x))\tilde{N}(dt,d\alpha)>-\infty, \end{aligned}$$

which contradicts our assumption, hence $\tau_e=\infty$ a.s. □

3. Extinction

In this section we investigate the stability of the uninfected equilibrium E^0.

Theorem 3.1. *If $\mathcal{R}_0\le 1$ then the uninfected equilibrium E^0 of (1.3) is globally stochastically asymptotically stable on $\mathbb{D}$.*

Proof. We consider the following function

$$V_1(x,y,z)=\frac{1}{2}(x-\frac{s}{\mu}+y)^2+k_1y+\frac{k_2}{2}z^2. \tag{3.1}$$